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INSTITUTE OF
DISTANCE & ONLINE LEARNING

MASTER OF COMPUTER APPLICATIONS

OPERATIONAL RESEARCH

MCA633

Self Learning Material

R101

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OPERATIONAL RESEARCH

MCA633

Anand Sharma

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Operational Research

Course Code: MCA633 **Credits: 3**

Course Objectives:

- To apply quantitative methods and techniques for effective decisions–making, model formulation and applications that is used in solving business decision problems.
- To provide a formal quantitative approach to problem solving and an intuition about situation.
- To develop mathematical skills to analyze and solve integer programming from a wide range of applications.

Syllabus

Unit 1 – Introduction: The Historical development, Nature, Meaning and Management Application of Operational research.

Unit 2 – OR Modeling Approach: OR Modeling, Its Principle and Approximation of O.R.Models, Main characteristics and phases.

Unit 3 – OR Modeling Solving Methods: General Methods of solving models, Scientific Methods, Scope, Role on Decision Making and Development of Operation Research in India.

Unit 4 – Introduction to Linear Programming: Linear Programming: Formulation, Graphical solution.

Unit 5 –Solving LPP- The Simplex Method 1: Simplex method and its flow chart.

Unit 6 – Solving LPP- The Simplex Method 2: Two-phase Simplex method, Degeneracy. Big-M Method.

Unit 7 – Duality: Definition of Dual Problem, General Rules for converting any Primal into its Dual Simplex method and its flow chart.

Unit 8 – Transportation Problems: The transportation problem Stream line simplex method, Stream line simplex method for the transportation problem.

Unit 9 – Assignment Problems: Assignment problem, a special algorithm for the assignment problem.

Unit 10 – PERT and CPM: Network representation, Critical path (CPM) computations and PERT networks.

Text Books:

1. Hamdy, T., 1996, 7th Ed., *"Operations Research – An Introduction"*, Delhi: Prentice-Hall.
2. Sharma, J.K., 2014, *"Operations Research"*, Delhi: Pearson Education.
3. Sharma, S.D., 2012, *"Operation Research"*, Meerut: Kedarnath & Ramnath Company.

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1. Sasieni, M., Yaspan, A., 1959, 1st Ed., *"Operations Research – Methods & Problems"*, New York: John Wiley & Sons Friedman.

CONTENTS

UNIT 1 INTRODUCTION

Structure:

1.0 Learning Objectives

After studying this unit, you will be able to:

- Define nature and meaning of Operation Research.
- Describe the management application of Operation Reasearch.

1.1 Introduction

With the increased complexities of life, the business has grown tremendously in different directions and hence there is a necessity to modify the organisations relevant to business. Decision making is not practised only in business areas. It is the process of life balance. Thus, decision making is a requirement for each one of us, the humans, at all times. With the size and complexities of business, high cost of labour, materials and machines attract the attention of all those in saddle to make business competitive. Competition has also reduced the availability of time for decision making. This puts pressure on the quantity and the quality of information gathered/required for day-to-day or strategic decisions. Day-to-day life decisions are taken by us based on our knowledge and experience (input information used for processing the decisions), but when problem becomes complicated with large input data, the analysis becomes complicated and hence an effective use of systematic approach is needed. This has created the necessity of scientific methods for decision making for business.

These methods are called Quantitative Methods, Operations Research, Decision Science, Management Science, System Analysis or Operation Analysis etc. Hence Operations Research can be understood as a scientific tool evolved as an aid or supporting help for the decision making process. This is extensively used in business, industry, government and defence organisations for solving complicated problems, where large sums of money or national prestige are at stake. The use of mathematical and physical sciences develops the strategy, methodology and techniques for decision making. Better the quality of input information, better will be the quality and utility of the decision. By using definite scientific techniques, the analysis of quality input information further enhances the quality of the decision.

The aspect of constant change in the operating environment needs be taken into account, while selecting these methods and techniques. Thus, dynamic equilibrium is essential in decision making, leading to the concept of sensitivity analysis of the decisions, so that the changes or errors in numerical values do not alter the decision drastically. If these do, then a control mechanism or change in the techniques becomes imperative.

1.2 Development of Operation Research

The advent of Operations Research (commonly known as OR) was Second World War. The name was also derived from its use for research on Military Operations during the war. Since strategic and tactical decisions during the war are very complicated with time horizon for such decisions being comparatively small, the necessity for a group analysis and use of mathematical, economic and statistical theories along with Engineering, Behavioural and Physical Sciences was felt and utilised. American and British groups worked on various research projects. Success and usefulness of these projects led to the development of various techniques for decision making and later the results prompted their uses in business applications and civilian problems.

Hence, when the war ended, an effort was made to apply the OR approach to other areas in business and industry.

In 1947, George B. Dantzig developed linear programming and Simplex Method. Later some more techniques for Statistical Quality Controls, Dynamic programming, queuing theory and inventory related techniques were developed before 1950's. After this research and development, OR was introduced in the curricula of various universities and techniques were formulated and used in areas such as engineering, public administration, applied mathematics, management, economics and computer usage areas. Journal OR Quarterly was published in 1950, whereas the journal of Operations Research Society of India (ORSA), named 'Operations Research' was published in 1953.

Around 1967-68, the area of OR was extended for use in various functional requirements including behavioural problems of Administration.

1.3 Meaning and Definitions

The applicability of OR being very wide, it is difficult to define it in a very concise form. However, various definitions are available and are given below for consideration.

Operational Research is the application of the methods of science to complex problems in the direction and management of large systems of men, machines, materials and money in the industry, business, government and defence. The distinctive approach is to develop a scientific model of the system incorporating measurement of factors such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management in determining its policy and actions scientifically."

– Operations Research Society, UK.

Operations Research is concerned with scientifically deciding how to best design and operate man-machines systems usually requiring the allocation of scarce resources.

– Operations Research Society, America

Operations Research has been described as a method, a set of techniques, a team activity, a combination of many disciplines, an extension of particular disciplines (mathematics, engineering, economics), a new discipline, a vocation, even a religion, It is perhaps some of all these things.

– S.L.Cook

Operations Research is a scientific approach to problems solving for executive management.

– H.M. Wagner

Operations Research is the systematic application of quantitative methods, techniques and tools to the analysis of problems involving the operation of systems.

– Daellenback and George

Operations Research utilises the planned approach (updated Scientific method) and an interdisciplinary team in order to represent complex financial relationships as mathematical models for the purpose of providing a quantitative basis for decision making and uncovering new problems for quantitative analysis. *– Thierauf and Klekamp*

Operations Research in the most general sense can be characterised as the application of scientific methods, techniques and tools, to problems involving the operations of a system so as to provide those in control of the operations with optimal solutions to the problems.

– Churchman, Ackoff and Arnoff

Operations Research is applied decision theory. It uses any scientific, mathematical, or logical means to attempt to cope with the problems that confront the executive, when he tries to achieve a through-going rationality in dealing with his decision problems. *– D.W. Miller and M.K. Stan*

OR is a scientific method of providing executive departments with a quantitative basis for decisions under their control. *– P.M. Morse and G.E. Kimball*

From all the definitions given above, it is evident that the emphasis is on the decision maker to use the scientific methods, techniques and tools to help him in reaching a well-defined decision, as ultimate decision taking lies with the decision maker. Some definitions also leave the impression that either the techniques used are very cumbersome and lengthy or these can be used only for complex problems and very large applications.

In fact, most of the OR techniques are simple and can be used without much mathematical compli-cations. Hence, managers at various levels need not be scared of using these techniques. In simple words, the definition given by Churchman seem more appropriate to the situation in real life. It states as follows:

"**Operations Research**" is the application of *scientific methods, techniques* and *tools* to *problems* involving the operations of *systems* so as to provide *those* in *control* of operations with *optimum solution* to the problems. – *Churchman*

The words italicised show the important ingredients of the O.R. i.e., decision maker, problem, systems, operations, optimum solutions by using scientific approach.

1.4 Applications of Operation Research

It is most widely used technique for large number of applications in business industry as well as in various other fields. Some of the applications are given below :

Defence

1. Transportation costs
2. Optimum weaponry system
3. Optimum level of force deployment

Finance

1. Profit planning
2. Investment Policy for maximum return
3. Investment risk analysis
4. Auditing.

Marketing

1. Travelling salesman cost
2. Plant locations
3. Media selection.

1.5 Summary

Day-to-day life decisions are taken by us based on our knowledge and experience (input information used for processing the decisions), but when problem becomes complicated with large input data, the analysis becomes complicated and hence an effective use of systematic approach is needed.

Better the quality of input information, better will be the quality and utility of the decision.

The advent of Operations Research (commonly known as OR) was Second World War. The name was also derived from its use for research on Military Operations during the war.

Around 1967-68, the area of OR was extended for use in various functional requirements including behavioural problems of Administration.

1.6 Key Words/Abbreviations

- **Activity:** Any individual operation, which utilizes resources and has an end and a beginning, is called. activity.
- **Function:** an activity that is natural to or the purpose of a person or thing.
- **Production:** the action of making or manufacturing from components or raw materials, or the process of being so manufactured
- **Limitation:** a limiting rule or circumstance; a restriction

1.7 Learning Activity

1. What were the post world War II factors so important to the development of OR. Find it.

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2. Write in your own words about OR with examples.

1.8 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. What is Operations Research? How do you use it in day to day decision making process?
2. Describe the origin and development of Operations Research? What were the controlling factors giving birth to OR?
3. Describe situations where OR techniques can be used.
4. (a) Discuss various phases in solving an OR problem.

   (b) Discuss scientific methods in OR. [Punjab University, B.Sc. (Mech.), 1982, 84]
5. Give any three useful definitions of Operations Research and explain them.
   [Meerut University, IPM, 1991; M.Sc. (OR), 1990]

B. Multiple Choice/Objective Type Questions

1. Operation Research is the ________________________.

   (a) National emergency (b) Combined efforts of talents of all fields

   (c) Economics and Engineering (d) Political problems
2. The person who coined the name Operations Research is _____________________.

   (a) Bellman (b) Newman

   (c) McClosky and Trefrhen (d) None of the above
3. The first step in solving Operations Research problem is ______________________.

   (a) Model building (b) Obtain alternate solutions

   (c) Obtain basic feasible solutions (d) Formulation of the problem.

4. The objective of Operations Research is ____________________.
   (a) Optimal utilization of existing resources
   (b) To find new methods of solving Problems
   (c) To derive formulas
   (d) To utilize the services of scientists
5. Operational research is a very powerful tool for ____________________.
   (a) Research
   (b) Decision making
   (c) Operations
   (d) None of these

Answers:

1. (b), 2. (c). 3. (d), 4. (a), 5. (b).

1.9 References

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2. Gupta M.P. and J.K. Sharma, 2nd Ed., 1997, “Operations Research for Management”, National Publishing House, New Delhi, .
3. Kapoor V.K., “Operations Research”, (Fifty Ed. Reprint), 1997, Sultan Chand & Sons.
4. Sharma J.K., 1997,“Operations Research — Theory and Applications”, Macmillan India Ltd., New Delhi.
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UNIT 2 OR MODELING APPROACH

Structure:

2.0 Learning Objectives

After studying this unit, you will be able to:

- Explain OR modeling.
- Describe the principle and approximation of OR models.
- Elaborate the characteristics and phases of OR models.

2.1 Introduction

Modellers frequently classify their model as being one type or another: "differential equation model", "matrix model", "individual-based model", and so on. Simile is capable of handling most of these model types - sometimes directly, sometimes with some recasting. The aim of this section is to consider some of the more common model types, and how they are handled in Simile. It should be remembered, however, that a model in Simile can combine what would normally be considered distinct and non-combinable modelling approaches. Thus, in Simile, it is quite possible to have a model that combines, for example, differential-equation, matrix, and individual-based modelling.

2.2 OR Modeling

When we represent a real life situation in some abstract form whether physical or mathematical, bringing out relationships of its important ingredients, we call it as model. Thus, model need not describe all the aspects of this situation, but it should signify and identify important factors and their inter-relationships to describe the total situation.

Due to the representation of important relationships of various significant parameters, the model can act as a very helpful decision tool and various complicated uncertainties can be structured for their analysis and effects on the situation. It can then help in many managerial decisions.

There are large number of models used in OR. Some of the basic types are described below:

Physical Models

When we utilise all forms of drawings, sketches, diagrams, groups or charts to describe a situation or problem specific in nature, we term them as physical models. Since relationships of the parameters are depicted in the pictorial form, it becomes easier to comprehend the problem and facilitates its analysis. Bar and pie charts, graphs and histograms etc. describe production forecast, manpower utilisation, cash/funds flow, with their levels of fluctuations during the time period under consideration. There are two types of physical models.

(a) **Iconic Models:** An icon is the depiction of an object as its image or likeness. It is a useful tool but the application in management problem areas is restrictive and narrow. In Engineering and Scientific areas, these are used extensively such as pilot models or mockups to test the operation of new theories and new products. Simulation techniques fall under this category, where full scale product need not be manufactured, but experiments can be conducted on miniature fully functional models under simulated conditions.

(b) **Analog Models:** These models are similar to Iconic Models but not the exact replica of the actual system. These are small physical systems having similar characteristics such as children toys, model cars and rail lines etc. The aim is to visualise how the system should look like and how routine procedures can be demonstrated such as Building models.

Symbolic Models

These models are used to represent actual problems. There are two types of symbolic model:

(a) **Verbal Models:** When we represent the problem and parameter inter-relationships written or spoken in words, these are called Verbal models. For example a magazine, a book or an advertisement is a verbal model.

(b) **Mathematical Models:** When we represent the problem and its parameters by a set of mathematical expressions, these are called Mathematical models. Though these are abstract, but have definite and precise manipulation possible under the laws of mathematics. These can be either deterministic or probabilistic models.

(i) **Deterministic models:** When the variables and their relationships can be defined precisely, for example, profit = Sales volume × Profit per unit.

It can be represented as

$$R = S \times P$$

where R = Profit earned

S = Sales volume

P = Profit per unit product

Similarly Total Cost = Fixed cost + Sales × Variable cost

or $TC = FC + n \,.\, VC$

where n = Sales Volume

FC = Fixed Cost

VC = Variable Cost per unit

(ii) Probabilistic models: When risks or uncertainties can be represented in mathematical relationship form, the models are called 'Probilistic models'. Thus, the outcome of a decision will be based when risk or uncertainty level is defined in proper assumption.

Heuristic Models

These models use intuititive rules or guidelines to solve a particular problem. These models are not based on any definite mathematical expressions or relationships, but problem solving based on past experience or approach formulated on the basis of definite stepped procedure. These models need an ample amount of creativity and experience by the decision maker.

2.3 Its Principles and Approximation of OR

OR can be used for any decision making and control functions using the following strategical approaches.

1. Methodical Approach: The Operations Research is a systematic application of scientific methods, principles, tools and techniques for a particular given problem in whatsoever field. These are used to obtain the optimal level of operations based on the available circumstances of the situation. The data, its analysis and results are to be implemented for its effectiveness, otherwise an alternate plan has to be worked out. The method is so systematic in application approach that it is easily termed as structural or methodical approach.

2. Objective Approach: Since the use of Operations Research is meant to find the optimal solution to a given problem, it is objective based in order to ensure its desirability for the organisational profitability.

3. Wholistic Approach: Since we examine the importance and relative correlation of all the objectives, howsoever conflicting or multiple in nature these may be this is called wholistic approach, because it takes care of and validates the claims of various sub-parts of an organisation. It can be called inter-disciplinary approach also for the same reason.

Though models are useful and helpful tools for the decision makers, the underlying assumption in their formulation becomes the limitation of the models. Since all the real life situations cannot be accurately quantified due to its dynamic nature, these models need be adjusted to the situation in question. However, the limitations can be summarised as follows:

1. Models are constructed based on certain assumptions and the nature of relevant factors. Since all the factors and their nature cannot always be predicted and quantified, the models remain valid under assumed conditions only.
2. Models are abstractions of real life and hence cannot replace the reality of life. Intangibility and dynamic nature of the parameters make the model unusable in all conditions.
3. If we formulate the model taking all possible factors into account, it will be too complex and unweildy for any useful business purpose. The time consumed on such solutions may not be worthwhile with respect to its utility as compared to time and cost involved.

2.4 Phases in the use of Operations Research

The approach to "Operations Research" can be divided into three logical phases.

1. **Judgement Phase:** The problem starts with the identification of the problem as faced in real life. The solution to the problem can be then directed towards the organisational objective. It will involve various variables related to the specific objective. It is only then that the application of an appropriate measure of its usefulness to the organisation can be formulated and put into structural form with relevant essential information for the decision maker.
2. **Research Phase:** In this phase, relevant data is collected for the problem related parameters so as to define and understand the problem in its entirety. This data is then put into use for formulation of appropriate model and then decide how to validate the result out of the given information, by data testing the hypothesis so selected. There may be requirement of additional data to test its applicability over a wide range of observations and variability. Based on the analysis and verification of the data, the usefulness or desirability of the method or model, the predictions can be made. The generalisation of results and consideration of alternative methods for 'what if' system is then standardised.

3. **Action Phase:** During this phase, the recommendations for the implementation of the decision so arrived are made by the person carrying out the analysis. This final recommendation has to be based on the actual problem and its reason for arising including the environment in which the problem occurred. Various assumptions, limitations and omissions for the objective need to be spelt out.

2.5 Summary

The model is a collection of logical and mathematical relationships that represents aspects of the situation under study. Models describe important relationships between variables, include an objective function with which alternative solutions are evaluated, and constraints that restrict solutions to feasible values. Models must be both tractable, capable of being solved, and valid, representative of the original situation. These dual goals are often contradictory and are not always attainable. It is generally true that the most powerful solution methods can be applied to the simplest, or most abstract, model.

2.6 Key Words/Abbreviations

- **Models:** A three-dimensional representation of a person or thing or of a proposed structure, typically on a smaller scale than the original.
- **Activities:** The condition in which things are happening or being done.
- **Manufacturing:** The making of articles on a large scale using machinery; industrial production.
- **Applications:** The action of putting something into operation.

2.7 Learning Activity

1. Write in your own words about OR modeling with examples.

----

2. List the types of OR models and its use.

2.8 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. What are the different types of models used in Operations Research? Explain in detail. [Rajasthan University, M.Com., 1983]
2. It is common for business to ensure against the occurrence of events which are subject to varying degree of uncertainty, for example, ill-health of senior executives. At the same time, the use of formal analytical models to assist in the process of making decisions on business problems which are generally subject to uncertainty does not appear to be very widespread.

   Describe the model building approach to the analysis of business problems under conditions of uncertainty. Discuss the apparent inconsistency in companies 'willingness to ensure when formal analytical models of an Operations Research nature will allow for uncertainty are relatively rarely employed. [I.C.M.A. (London), Nov., 1981]
3. Describe the phases used in OR?

B. Multiple Choice/Objective Type Questions

1. Operation research is the use of _______________.

   (a) mathematical models (b) Statistics

   (c) algorithm to aid in decision-making (d) All of these

2. Operation Research models in which some or all variables are random in nature are called ___________.

   (a) Physical models (b) Probabilistic models

   (c) Symbolic models (d) Deterministic models

3. Operation Research models in which values of all variables and all possible outcomes are known with certainty are called ______________.

   (a) Physical models (b) Symbolic models

   (c) Deterministic (d) Probabilistic models

4. Operation Research uses models to help the management to determine its _________ scientifically.

   (a) Policies (b) Actions

   (c) Both A and B (d) None of these.

5. A solution may be extracted from a model either by ____________.

   (a) Conducting experiment on it (b) Mathematical analysis

   (c) Both A and B (d) None of the above.

Answers:

1. (d), 2. (b), 3. (c), 4. (c), 5. (c).

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UNIT 3 OR MODELING SOLVING METHODS

Structure:

3.0 Learning Objectives

After studying this unit, you will be able to:

- Explain general methods of solving models.
- Describe scientific methods.
- Illustrate the role of decision making and development of OR in India.

3.1 Introduction

The Mathematical programming involves optimisation of a certain function, called *objective function,* subject to the given limitations or *constraints.* A manager may be faced with the problem of deciding the appropriate product mix taking the objective function as the maximising of profits obtainable from the mix, keeping in view various constraints such as availability of raw materials, position of labour supply, market consumption etc.

The linear programming method is a technique of choosing the best alternative from a set of feasible alternatives in situations in which the objective function as well as constraints can be expressed as linear mathematical function.

3.2 General Methods of Solving Models

Situations available for decisions making are as under:

1. Quality of decision will be superior, if all aspects of a system are known.
2. Normally, all factors are not certain. So chances of decisions are found out where certain risks are involved and have to be lived with.
3. When nothing is known about a system, the outcome decision is likely to be inferior due to risks and uncertainty.

Hence, we can consider decision making under three situations:

(a) Decision making under certainty

(b) Decision making under risks and conflicts

(c) Decision making under uncertainty.

The models formulated to solve a particular problem should be very simple, but capable of giving desired result for the help of the decision-maker. In any case, the quality of decision making depends on the quality of input information data and its logical application and analysis towards a given objective. Hence, over simplification of the model at the cost of its purpose should not be tried out.

OR models can be classified in the following categories:

Linear Programming Models: When decision making pertains to profits, cost etc. and these parameters have a linear relationship of several variables, the model is known as Linear Programming Model having constraints or limitations on various resources also as linear function of the decision variables or parameters. The constraints in linear form can be expressed either as equalities or inequalities.

When the decision variables values are required to be integers, such as number of men, machines etc. the constraint of such a nature can be inbuilt into the model and this technique is called Integer Programming.

The problems, which have multiple, conflicting and incommensurable objective functions but under linear constraints, the model is termed as Goal Programming Model. When the decision variables are not definite or deterministic, but depend on the chance, the problem becomes stockastic goal programming problem.

For working out the cost and time minimisation based on deterministic information available for unit cost or cost per unit distance, the model formulation is called Transportation Problem. When definite resources are allocated to perform certain assigned activities, such problems are called Assignment problems and models so used are called Assignment Models.

Sequencing Models: Instead of assigning the jobs in a definite activity system, when we have to determine in what sequence the activities should be performed out of given resources in the most cost/time effective manner, the models are called Sequencing Models.

Waiting Line or Queuing Models: These models are used to establish a trade off between the cost of waiting of customer and that of providing service following a queue system. In this case, we have to describe various components of the system such as traffic intensity, average waiting time of the customer in the queue, average queue length, etc.

Games Models: These models are formulated and utilised to describe the behaviour of two or more opponents or players who are performing the functions to achieve certain objectives or goals and in the bargain, would gain or loose in the business process. Such models are very effectively used for optimising strategies of the players with respect to the anticipated strategies of the competing players.

Dynamic Programming Models: These models are the offshoots of the mathematical programming for optimising the multistage decision processes. The problems are solved by first dividing the problem into sub-problems or stages and solving them sequencially till the original problem has been solved.

Inventory Models: These models are primarily meant for working out optimal level of stocking and ordering of items for a given situation. Main objective is to optimise the cost under conflicting requirements of ordering, holding and shortages. Quantity discounts and selective inventory controls are also useful derivations.

Replacement Models: These models are utilised when we have to decide the replacement policy for an equipment for one reason or the other. The deterioration of efficiency of the equipment with use and time is the reason for such replacement whether partial or full. The differing performance parameters create variations in the form of varied replacement policies with the help of Replacement Models.

Simulation Models: These models are utilised when we want to evaluate the merit of alternate course of action by experimenting with a mathematical mode of the problem and the variables in the problem are random. Thus, repetition of the process by using simulation model provides an indication of the merit of the alternative course of action with respect to the decision variables.

Network Models: These are basically project management models utilised in planning, monitoring and controlling various projects, where utilisation of human and non-human resources has to be optimised with reference to the time and cost available for the project. CPM/PERT as basic network models help in identification of important bottlenecks or potential trouble areas. The techniques improve the project co-ordination by working out effective trade-off analysis for the resource allocations.

Decision Analysis Models (Decision Theory): These models are used for selection of optimal strategy of operation given the possible pay offs and their associated probabilities of occurrence. The models are used for decision making process under uncertainty or risk conditions.

Hence, these models can be broadly classified as follows:

(i) Deterministic Models

(ii) Probabilistic Models

(iii) Heuristic Models.

3.3 Scientific Activity

The scientific method has five basic steps, plus one feedback step:

- Make an observation.
- Ask a question.
- Form a hypothesis, or testable explanation.
- Make a prediction based on the hypothesis.
- Test the prediction.
- Iterate: use the results to make new hypotheses or predictions.

3.4 Scope of Operations Research in Management

The techniques used in Operations Research have very wide application in various fields of business/industrial/government/social sector. Few areas of applications are mentioned below:

Marketing and Sales

1. Product selection and competitive strategies.
2. Utilisation of salesmen, their time and territory control, frequency of visits in sales force analysis.
3. Marketing advertising decisions for cost and time effectiveness.
4. Forecasting and decision trends.
5. Pricing and competitive decisions.
6. Market research decisions.

Production Management

1. Product mix and product proportioning.
2. Facility and production planning and scheduling/sequencing
3. Physical distribution, warehousing and retail outlets planning—nature and localities.
4. Material handling facilities planning.
5. Assembly line balancing.
6. Maintenance policies and crew sizing and replacement systems.
7. Project planning, scheduling, allocation of resources, monitoring and control systems.
8. Design of Information Systems.
9. Queuing system design.
10. Quality control decisions.

Purchasing, Procurement and Inventory Controls

1. Buying policies levels and prices
2. Negotiation and bidding policies
3. Time and quantity of procurement.

Defence

1. Optimum weaponry systems
2. Optimum level of force deployment
3. Transportation costs
4. Assignment suitabilities.

Finance, Investments and Budgeting

1. Profit planning
2. Cash flow analysis

3. Investment policy for maximum return
4. Dividend policies
5. Investment decisions and risk analysis
6. Claim and compensation procedures
7. Portfolio analysis.

Personnel Management

1. Determination of optimum organisational level
2. Job evaluation and assignment analysis
3. Mixes of age and skills
4. Salary criteria
5. Recruitment policies and job description.

Research and Development

1. Determination of areas of thrust for research and development
2. Selection criteria for specific project
3. 'What if' analysis for alternative design and reliability
4. Trade-off analysis for time-cost relationship and control of development projects.

3.5 Role on Decision Making

Decision making is an everyday process often without the aid of pencil or paper. It is a major role of any manager. Most of the decisions taken govern the fortunes of business-right decisions having salutary effect and wrong ones creating disaster. Decisions having direct/indirect bearing on finances need, therefore, better deliberations.

Decisions may be tactical or strategic in nature, tactical decision affecting the business in short run, whereas strategic ones having far reaching effects like launching new products, expansion or diversification etc.

Normally tactical decisions tend to be taken quickly based on past experience for quick action. But a good manager should ponder over the problem, identify and specify the problem, collect enough information for decision making, its analysis and weighting various alternate possible solutions. Quantum of data and depth of analysis including time available for decision making would indicate the quality of decision.

3.6 Development of Operation Research in India

In 1953, The Institute of Management Sciences (TIMS) was established and its Journal 'Management Science' appeared in 1954. In our country, an OR establishment was formed at Regional Research Laboratory (RRL) at Hyderabad in 1949. In 1953, an OR team was formed at Calcutta whereas Formal OR Society of India was founded in 1957. Its journal is named 'OPSEARCH'. In 1953 itself, India became the Member of the International Federation of Operations Research Societies (IFORS) with its Headquarter in London. In 1954, some project scheduling techniques such as CPM and PERT, so commonly used now, were developed.

3.7 Summary

Most operations research studies involve the construction of a mathematical model. The model is a collection of logical and mathematical relationships that represents aspects of the situation under study. Models describe important relationships between variables, include an objective function with which alternative solutions are evaluated, and constraints that restrict solutions to feasible values.

3.8 Key Words/Abbreviations

- **Scope:** The opportunity or possibility to do or deal with something.
- **Role:** The function assumed or part played by a person or thing in a particular situation.
- **Development:** The process of developing or being developed.
- **Decision making:** The action or process of making important decisions.

3.9 Learning Activity

1. What do you mean by development of operation research in india?

   ----

2. List the OR models classifications.

3.10 Unit End Questions (MCQ and Descriptive)

A. Descriptive Type Questions

1. Explain the concept, scope and tools of OR as applicable to business and industry.
2. What is the role of OR in decision making?

B. Multiple Choice/Objective Type Questions

1. The first country to use Operations Research method to solve problems is ______________.

   (a) India (b) China

   (c) U.K. (d) U.S.A

2. The problem, which is used to disburse the available limited resources to activities, is known as ________________________.

   (a) Allocation Model (b) O.R. Model

   (c) Resources Model (d) Activities model

3. One of the properties of Linear Programming Model is _______________________.

   (a) It will not have constraints

   (b) It should be easy to solve

(c) It must be able to adopt to solve any type of problem,

(d) The relationship between problem variables and constraints must be linear

4. Operation research attempts to find the best and _______ solution to a problem

   (a) Optimum (b) Degenerate

   (c) Perfect (d) None of these

5. What have been constructed for operation research problems and methods for solving the models that are available for many cases?

   (a) Scientific models (b) Algorithms

   (c) Mathematical models (d) None of these

Answers:

1. (c), 2. (a), 3. (d), 4. (a), 5. (c).

3.11 References

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UNIT 4 INTRODUCTION TO LINEAR PROGRAMMING

Structure:

4.0 Learning Objectives

After studying this unit, you will be able to:

- Explain the characterstics and uses of linear programming.
- Describe the assumptions underlying linear programming.

- Explain the advantages and applications of linear programming.
- Elaborate the Linear programming model formulation procedure.
- Utilisation of linear programming model formulation for business requirements.
- Assess through self-assessment problems.
- Define formulation of linear programming.
- Describe graphical solution to linear programming.

4.1 Introduction

Formulation of linear programming is the representation of problem situation in a mathematical form. It involves well defined decision variables, with an objective function and set of constraints.

4.2 Requirements for Application of Llinear Programming

1. The aim or object should be clearly identifiable and definable in mathematical terms. For example, it would be optimisation of either cost or profits or time etc.
2. The activities involved should be distinct and measurable in quantitative terms such as products involved in a production planning problem.
3. The resources to be allocated also should be measurable quantitatively. Limited availability/constraints should be clearly spelt out.
4. The relationships representing the objective function as also the resource limitation consideration must be linear in nature.
5. There should be a series of feasible alternative courses of action available to the decision maker, that are determined by the resources constraints.

4.3 Assumptions underlying Linear Programming

1. **Proportionality** – Basic assumption of LP is that proportionality exists in the objective function and the constraints inequalities. This means that the amount of each resource

used and associated contribution to profit or cost in the objective function should be optimal, proportional to the value of each decision variable. If we increase the production quantity, the resources requirement should also be increased in the same proportion.

2. **Additivity –** It indicates that in the objective function and constraint inequalities both, the total of all the activities is given by the sum total of each activity conducted separately. Thus total profitability and sum total of all resources required should be equal to the sum of the individual amounts.
3. **Continuity –** It is also an assumption of a linear programming model that the decision variables are continuous. As a consequence, combinations of output with fractional values, in the context of production problems, are possible and obtained frequently. Normally we deal with integer values, but even fractional values can be utilised. Fractional values should be considered only for one time decision problems.
4. **Certainty –** Various parameters namely the objective function coefficients, the coefficients of inequality/equality constraints and the constraints (resource) values are known with certainty. Hence, linear programming is deterministic in nature.
5. **Finite Choices –** A linear programming model also assumes that a limited number of choices are available to the decision maker and the decision variables can also assume negative values.

4.4 Advantages of Linear Programming

1. Linear programming is a useful technique to obtain optimum use of productive resources. It helps a decision maker to ensure effective use of scarce resources by their proper deployment.
2. Due to its structured form, linear programming techniques improve the quality of decision making.
3. It generates large number of alternate solutions and hence it helps in reaching practical solutions at optimum working level. It also permits modifications of the mathematical model to suit the decision makers requirement.

4. This technique also indicates ideal capacity of machines or materials in a production process. In fact it helps decision maker to decide whether his resources can be intentionally kept idle in order to work on optimal level of objective, if certain constraints demand so.
5. This technique can also cater for changing situations. The changed conditions can be used to readjust the plan decided for execution.

4.5 Formulation of LP Problems

From the above, we can establish a vast area of applicability of the LP technique. But to get the best advantage out of the process, we have to clearly identify objective function, decision variables and constraints and quantify the relationship to formulate a worth-while mathematical model. There are three basic steps in formulation of linear programming model.

Step 1: Identify the decision variables to be determined. These should be brought into algebraic relation form for utilisation.

Step 2: Clearly define all the limitations for a given situation. These limitations or constraints also need to be expressed in algebraic form either as linear equations or inequalities in terms of the decision variables so identified in step 1.

Step 3: Identify the objective to be optimised and it also should be expressed in terms of linear function of decision variables.

The formation of the problem now can be achieved in a very structured form by bringing in all related combinations.

In todays environment, the quote below is very valid.

4.6 Mathematical Model Formulation of LP Problem

In order to develop a general procedure for solving any linear programming (LP) problem, we first introduce the standard form. Let us assume the decision variables as x_1, x_2, x_3,.... x_n such that the objective function (Linear) of these variables assumes an optimum value, when operated under the given constraint of resources. Thus, the standard form of LPP can be written as follows.

Objective Function

Optimise (Maximise or minimise) $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots\ldots\, c_n x_n$. where c_j (j = 1,2,...................n) are called cost coefficients.

Constraints (Linear)

Subject to,

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, a_{2n} x_n = b_2$

....

....

....

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, a_{mm} x_n = b_m$.

Where b_i (i = 1, 2...................m) are resources constraints and constants aij (i = 1,2,.........................m; j = 1,2,..................n) are called the input output coefficients

Also,

$x_1, x_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\, x_n \geq 0$ (non-negative constraint)

The decision variables are required to be non-negative so that they can contribute towards the optimum objective function, which is either maximisation or minimisation type.

4.7 Solved Examples

Problem 1:

A company is planning to manufacture two products in a manufacturing unit. The products are Radios and TV's. Both the products have four distinct departments to pass through, i.e., chassis, cabinet, assembly and testing. Monthly capacities of each department are given in the matrix below:

	TVs	*Radios*
Chassis	1,500	4,500
Cabinet	1,000	8,000
Assembly	2,000	4,000
Testing	3,000	9,000
Unit Profit	₹ 300	₹ 50
Sales Limit	11,000	Unlimited

Solution:

For formulation of the above problem into a linear programming model, we can proceed as follows:

1. Identify objectives – Maximisation of profit.
2. Identify decision variables -
   - Number of TVs to be manufactured say x_1.
   - Number of Radios to be manufactured say x_2.
3. Identify Constraints –
   - Production capacity of each department i.e., Chassis, Cabinet, Assembly and Testing.
   - Sales limitations.
   - Non-negative and integer characteristics of the decision variable.

These can be converted into mathematical form as under:

Objective Function

Maximise profit (Z) = $300x_1 + 50x_2$ (Total profit for x_1 TVs and x_2 Radios)

Constraints

$$\frac{x_1}{1,500} + \frac{x_2}{4,500} \leq 1 \qquad \text{(Chassis Constraint)}$$

$\frac{x_1}{1,000} + \frac{x_2}{8,000} \leq 1$ (Cabinet Constraint)

$\frac{x_1}{2,000} + \frac{x_2}{4,000} \leq 1$ (Assembly Constraint)

$\frac{x_1}{3,000} + \frac{x_2}{9,000} \leq 1$ (Testing Constraint)

and $x_1 \leq 11,000$ (Sales Constraint)

also, $x_1 \geq 0$ Non-negative integers as

$x_2 \geq 0$ TVs and Radios are to be definite positive numbers

Problem 2:

A manufacturing unit has three products on their production line. The production capacity for each product is 50, 30 and 45 respectively. The limitation in the production shop is that of 300 manhours as total availability and the manufacturing time required per product is 0.5, 1.5 and 2.0 manhours. The products are priced to result in profits of ₹ 10, 15 and 20 respectively. If the company has a daily demand of 25 units, 20 units and 35 units for respective products, formulate the problem as LP model so as to maximise the total profit.

Solution:

The information available can be put into the structural matrix form as follows :

Requirement	*Product*			*Total*
	P	*Q*	*R*	
Production Capacity	50	30	45	—
Production manhours per unit	0.5	1.5	2.0	300
Profit per unit	10	15	20	—
Daily demand	25	20	35	—

Let the number of units to be manufactured be x_1, x_2 and x_3 respectively for product *P, Q* and *R*.

Then decision variables are related to the profit related information.

Hence, Maximise Profit = $10x_1 + 15x_2 + 20x_3$.

The given constraints are that of production capacity, manhours availability for production and daily demands. These can be converted into linear relations as follows:

$$0.5x_1 + 1.5x_2 + 2.0x_3 \leq 300$$

$$\left.\begin{aligned} x_1 &\leq 50 \\ x_2 &\leq 30 \\ x_3 &\leq 45 \end{aligned}\right\} \quad \text{Resources Constraints}$$

$$\left.\begin{aligned} x_1 &\geq 25 \\ x_2 &\geq 20 \\ x_3 &\geq 35 \end{aligned}\right\} \quad \text{Demand Constraints}$$

and x_1, x_2 and x_3 are to be non-negative integer values, the products being produced in whole numbers.

Problem 3:

An electronics company produces three types of parts for automatic washing machine. It purchases castings of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines. The selling prices of part A, B and C respectively are ₹ 8, ₹ 10 and ₹ 14. All parts made can be sold. Castings for parts A, B and C respectively cost ₹ 5, ₹ 6 and ₹ 10.

The shop possesses only one of each type of machine. Costs per hour to run each of the three machines are ₹ 20 for drilling, ₹ 30 for shaping and ₹ 30 for polishing. The capacities (parts per hour) for each part on each machines are shown in the following table.

Machine	*Capacity per hour*		
	Part A	*Part B*	*Part C*
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximise profit for an hour's run. Formulate this problem as an LP model.

[*Delhi University, M.B.A., 1986*]

Solution:

The decision variables in this case are clearly the number of parts of each type to be produced by the shop per hour of run.

Let these be x_1, x_2, x_3 for parts A, B and C respectively.

The aim is to maximise the profit. Hence the objective function will be to maximise profit.

profit for part A = Selling price – Cost price – Manufacturing cost

$$=(8-5)-\left[\frac{20}{25}+\frac{30}{25}+\frac{30}{40}\right]= ₹\ 0.25$$

Similarly unit profit for part B $=(10-6)-\left[\frac{30}{40}+\frac{30}{20}+\frac{30}{30}\right]= ₹\ 1.00$

and unit profit for part C $=(14-10)-\left[\frac{20}{25}+\frac{30}{20}+\frac{30}{40}\right]= ₹\ 0.95$

Hence, Max. Profit $= 0.25x_1 + 1.00x_2 + 0.95x_3$.

Now the constraints can be described as foll ows

$$\frac{x_1}{25}+\frac{x_2}{40}+\frac{x_3}{25}\leq 1$$ Drilling constraint

$$\frac{x_1}{25}+\frac{x_2}{20}+\frac{x_3}{20}\leq 1$$ Shaping constraint

$$\frac{x_1}{40}+\frac{x_2}{30}+\frac{x_3}{40}\leq 1$$ Polishing constraint

and $x_1, x_2, x_3 \geq 0$ Non-negative constraint

Problem 4:

ABC company manufactures three grades of paints, Venus, Diana and Aurora. The plant operates on a three-shift basis and the following data are available from the production records.

Requirement of resources	*Grade*			*Availability*
	Venus	*Diana*	*Aurora*	*(capacity per month)*
Special additive (kg/litre)	0.30	0.15	0.75	600 tonnes
Milling (kilo-litre per machine shift)	2.00	3.00	5.00	100 machine shifts
Packing (kilo-litres per shift)	12.00	12.00	12.00	80 shifts

There are no limitations on other resources. The particulars of sales forecasts and estimated contribution to overheads and profits are given in the following:

	Venus	*Diana*	*Aurora*
Max. possible sales per month (KL)	100	400	600
Contribution (₹ per KL)	4,000	3,500	2,000

Due to commitments already made, a minimum of 200 kilo-litres (KL) per month of Aurora has to be necessarily supplied in the next year.

Just as the company was able to finalise the monthly production programme for the next 12 months, an offer was received from a nearby competitor for hiring 40 machine shifts per month of Milling capacity for grinding Diana paint, that could be spared for at least a year. However, due to additional handling, the profit margin of the competitor involved, by using this facility, the contribution from Diana will get reduced by ₹ 2 per litre.

Formulate this problem as an LP model for determining the monthly production programme to maximise contribution.

[*Delhi University, M.B.A., 1989*]

Solution:

Let x_1 = Quantity of Venus produced in KL

x_2 = Quantity of Diana produced in KL

x_3 = Quantity of Diana from hired facility in KL

x_4 = Quantity of Aurora produced in KL

$\therefore$ Max. Profit $= 4{,}000x_1 + 3{,}500x_2 + (3{,}500 - 2{,}000)x_3 + 2{,}000x_4$

Now we work out the constraints relationship as follows

$0.30x_1 + 0.15x_2 + 0.15x_3 + 0.75x_4 \leq 600$	Special additive constraint
$\frac{x_1}{2} + \frac{x_2}{3} + \frac{x_4}{5} \leq 100$	Milling capacity (internal) constraint
$\frac{x_3}{3} \leq 40$	Milling capacity (externally hired) constraint
$\frac{x_1}{12} + \frac{x_2}{12} + \frac{x_3}{12} + \frac{x_4}{12} \leq 80$	Packing constraint
$x_1 \leq 100$	(for Venus)
$x_2 + x_3 \leq 400$	(for Diana) Marketing constraint
$200 \leq x_4 \leq 600$	(for Aurora)
$x_1;\ x_2;\ x_3;\ x_4 \geq 0$	Non-negative constraints

Problem 5:

A company is making two products A and B. The cost of producing one unit of product A and B is ₹ 60 and ₹ 80 respectively. As per the agreement, the company has to supply at least 200 units of product B to its regular customers. One unit of product A requires one machine hours whereas product B has machine hours available abundantly within the company. Total machine hours available for product A are 400 hours. One unit of each product A and B requires one labour hour each and total of 500 labour hours are available. The company wants to minimise the cost of production by satisfying the given requirements. Formulate the problem as a linear programming problem.

Solution:

Let us have the manufacture of x_1 and x_2 units of product A and B respectively. Then the given data indicates the relationship of decision variables as follows:

Objective function Minimise cost $= 60x_1 + 80x_2$ (production cost)

Constraints; $x_2 \geq 200$ (agreement for supply)

$x_1 \leq 400$ (machine hours for product A)

$$x_1 + x_2 \leq 500 \quad \text{(labour hours)}$$

and $$x_1, x_2 \geq 0 \quad \text{(non-negative constraint)}$$

Problem 6:

The owner of Metro Sports wishes to determine how many advertisements to place in the selected three monthly magazines A, B and C. His objective is to advertise in such a way that total exposure to principal buyers of extensive sports goods is maximised. Percentage of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used:

Requirement	*Magazines*		
	A	*B*	*C*
Readers	1 lakh	0.60 lakhs	0.40 lakhs
Principal buyers	15%	15%	7%
Cost per advertisement (₹)	5,000	4,500	4,250

The budgeted amount is at the most ₹ 1 lakh for advertisements. The owner has already decided that magazine A should have no more than 6 advertisement and B and C each have at least two advertisements. Formulate an LP model for the problem.

Solution:

Let us denote x_1, x_2 and x_3 as the number of advertisement in magazines A, B and C respectively. As per the problem, the objective of the owner is to have maximum exposure. Hence

Objective function;

Maximise Exposure = (15% of 1 lakh) x_1 + (15% of 0.60 lakh) x_2 + (7% of 0.40 lakh) x_3

Constraints are,

$$5{,}000x_1 + 4{,}500x_2 + 4{,}250x_3 \leq 1{,}00{,}000$$

$$x_1 \leq 6$$

$$x_2 \geq 2$$

$$x_3 \geq 2$$

and $$x_1;\ x_2;\ x_3 \geq 0$$

Problem 7:

A firm assembles and sells two different types of outboard motor A and B; using four resources. The production process can be described as follows:

Resources	*Capacity per month*
Motor unit shop resource	400 type A units or 250 type B units or any linear combination of the two.
Type A gear and drive shop resource	175 type A units
Type B gear and drive shop resource	225 type B units
Final assembly resource	200 type A units or 350 type B units or any linear combination of the two.

Type A units bring in a profit of the ₹ 90 each and type B units ₹ 60 each. What should be the optimum product mix? Formulate the problem.

Solution:

Let there be x_1 and x_2 units sold for type A and B units respectively.

Then as per problem, to maximise the profit,

Max profit (Ƶ) = $90x_1 + 60x_2$

and constraints are

$$\frac{x_1}{400} + \frac{x_2}{250} \leq 1$$

$$x_1 \leq 175$$

$$x_2 \leq 225$$

$$\frac{x_1}{200} + \frac{x_2}{350} \leq 1$$

$$x_1;\ x_2 \geq 0$$

Problem 8:

Four products have to be processed through a plant, the quantities required for the next production period being

product 1: 2,000 unitsproduct 2: 3,000 units

product 3: 3,000 unitsproduct 4: 6,000 units

There are three production lines on which the products could be processed. The rate of production in units per day and the total available capacity in days are given in the following table. The cost of using the lines is ₹ 600, 500 and 400 per day respectively.

Production Line	*Product*				*Maximum*
	1	*2*	*3*	*4*	*Linedays*
1	150	100	500	400	20
2	200	100	760	400	20
3	160	80	890	600	18
Total	2,000	3,000	3,000	6,000	

Formulate the problem as an LP model to minimise the cost of operation.

Solution:

Let x_{ij} = number of units of product i (i = 1, 2, 3, 4) produced on production line j; (j = 1, 2, 3), per day. Then

$$\text{Minimise total operation cost } Z = 600\sum_{i=1}^{4} x_{i1} + 500\sum_{i=1}^{4} x_{i2} + 400\sum_{i=1}^{4} x_{i3}$$

subject to,

$$\sum_{j=1}^{3} x_{1j} = 2{,}000$$

$$\sum_{j=1}^{3} x_{2j} = 3{,}000$$

$$\sum_{j=1}^{3} x_{3j} = 3{,}000$$

$$\sum_{j=1}^{3} x_{4j} = 6{,}000$$

and $$\frac{x_{11}}{150} + \frac{x_{21}}{200} + \frac{x_{31}}{500} + \frac{x_{41}}{400} \le 20$$

$$\frac{x_{12}}{200} + \frac{x_{22}}{100} + \frac{x_{32}}{760} + \frac{x_{42}}{400} \le 20$$

and $$\frac{x_{13}}{160} + \frac{x_{23}}{80} + \frac{x_{33}}{890} + \frac{x_{43}}{600} \le 18$$

with $x_{ij} \ge 0$ for all values of *i* and *j*.

Problem 9:

A co-operative farm owns 100 acres of land and has ₹ 25,000 in funds available for investment. The farm members can produce a total of 3,500 manhours worth of labour during the months of September-May and 4,000 manhours during June-August. If any of these manhours are not needed, some members of the farm will use them to work on a neighbouring farm for ₹ 2 per hour during September-May and ₹ 3 per hour during June-August. Cash income can be obtained from the three main crops and two types of livestock, dairy cows and laying hens. No investment funds are needed for the crops. However, each cow will require an investment outlay of ₹ 3,200 and each hen will require ₹ 15.

However, each cow will require 1.5 acres of land, 100 manhours during summer. Each cow will produce a net annual cash income of ₹ 3,500 for the farm. The corresponding figures for each hen are no acreage, 50 manhours, 0.6 manhours during Sept.-May; 0.4 manhours during June-Aug. and an annual net income of ₹ 200. The chicken house can accommodate a maximum of 4,000 hens and the size of cattle shed limits the members to a maximum of 32 cows. Estimated manhours and income per acre planned in each of the three crops are:

		Paddy	*Bajra*	*Jowar*
Man hours:	Sept-May	40	20	25
	June-August	50	35	40
Net annual cash income (₹)		1,200	800	850

The co-operative farm wishes to determine how much acreage should be planted in each of the crops and how many cows and hens should be kept to maximise its net cash income.

Solution:

Objective Function:

Max. (net cash income) = $3{,}500x_1 + 200x_2 + 1{,}200x_3 + 800x_4 + 850x_5 + 2x_6 + 3x_7$

wherex_1, x_2 are number of cows and hens respectively.

x_3, x_4, x_5 are average acreage for paddy, bajra and jowar crops,

x_6 = extra manhours utilised during Sept-May

andx_7 = extra manhours utilised during June-Aug.

Constraints

Manpower Constraint $100x_1 + 0.6x_2 + 40x_3 + 20x_4 + 25x_5 + x_6 \leq 3{,}500$ (Sept.-May)

$50x_1 + 0.4x_2 + 50x_3 + 35x_4 + 40x_5 + x_7 \leq 4{,}000$ (June-Aug.)

Land Constraint $1.5x_1 + x_3 + x_4 + x_5 \leq 100$

Livestock Constraint $x_1 \leq 32$ (cows)

$x_2 \leq 4{,}000$ (hens)

Investment Constraint

$$3{,}200x_1 + 15x_2 \leq 25{,}000$$

and $x_1; x_2; x_3; x_4; x_5; x_6; x_7 \geq 0$

Problem 10:

Ex-servicemen Airport Services company is considering the purchase of new vehicles for the transportation between Delhi airport and hotels in the city. There are three vehicles under consideration: station wagons, minibuses and large buses. The purchase price would be ₹ 1,45,000 for each station wagon; ₹ 2,50,000 for the minibus and ₹ 4,00,000 for large buses each. The Board of Directors has authorised a maximum amount of ₹ 50 lakhs for these purchases. Because of the heavy air travel, the new vehicles would be utilised at maximum capacity regardless of the type of vehicles purchased.

The expected net annual profit would be ₹ 15,000 for the station wagon; ₹ 35,000 for the minibus and ₹ 45,000 for the large bus. The company has hired 30 new drivers for the new vehicles. They are qualified drivers for all the three types of vehicles. The maintenance department has the capacity to handle an additional 80 station wagons. A minibus is equivalent to $1\frac{2}{3}$ station wagons and each large bus equivalent to 2 station wagons in terms of their use of the maintenance department. Determine the optimal number of each type of vehicle to be purchased in order to maximise profit.

Solution:

Let x_1, x_2 and x_3 be the number of station wagons, minibuses and large buses to be purchased.

As per the ratio of utilisation of maintenance department

1 minibus = $\frac{5}{3}$ station wagons and 1 large bus = 2 station wagons

Hence, objective function:

$$\text{Max. (Profit) } Z = 15{,}000x_1 + 35{,}000x_2 + 45{,}000x_3$$

$$\text{Subject to,} \quad x_1(1{,}45{,}000) + x_2(2{,}50{,}000) + x_3(4{,}00{,}000) \leq 50{,}00{,}000$$

$$\text{and} \quad x_1 + x_2 + x_3 \leq 30$$

$$x_1 + \frac{5}{3}x_2 + 2x_3 \leq 80$$

$$\text{and} \quad x_1;\ x_2;\ x_3 \geq 0$$

4.8 Self Assessment Problems

1. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the input and output per production run are given as follows:

Process	*Input*		*Output*	
(Units)	*Crude A*	*Crude B*	*Gasoline X*	*Gasoline Y*
1	5	3	5	8
2	4	5	4	4

The maximum amount available of crude A and B are 200 units and 150 units respectively. Market requirements show that atleast 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run from process 1 and process 2 are ₹ 300 and ₹ 400 respectively. Formulate this problem as a linear programming model.

2. A company makes three products X, Y and Z, which flow through three departments, Drill, lathe and Assembly. The hours of department time required by each of the products, the hours available in each of the department and the profit contribution of each of the products are given in the following table:

	Time required per unit			*Profit*
Products	*Drill*	*Lathe*	*Assembly*	*contribution/unit*
X	3	3	8	₹ 9
Y	6	5	10	₹ 15
Z	7	4	12	₹ 20
Hours available	210	240	260	

The marketing department of the company indicates that the sales potential for product X and Y are unlimited, but for product Z, it is only 30 units. Determine the optimal production schedule.

3. An investor has money making activities A_1, A_2, A_3 and A_4. He has only one lakh rupees to invest. In order to avoid excessive investment, no more than 50% of the total investment can be placed in activity A_1 and/or activity A_3. Activity A_1 is very conservative while activity A_4 is speculative. To avoid excessive speculation, atleast Re. 1 must be invested in activity A_1 for every ₹ 3 invested in activity A_4. The data on the return on investment are as follows:

Activity	*Anticipated return on investment (%)*
A_1	10
A_2	12
A_3	14
A_4	16

The investor wishes to know how much to invest in each activity to maximise the total return on the investment.

4. The Omega Data Processing company performs three types of activity: pay rolls, account receivables and inventories. The profit and time requirement for keypunch computation and office printing for a "standard job" are shown in the following table

	Profit/standard job	*Time Requirement (Min.)*		
Job	(₹)	Keypunch	Computation	Print
Payroll	275	1,200	20	100
A/c receivables	125	1,400	15	60
Inventory	225	800	35	80

Omega guarantees overnight completion of the job. Any job scheduling during the day can be completed during the day or night. Any job scheduled during the night, however, must be completed during the night. The capacity for both day and night are shown in the following table:

Capacity (min)	*Keypunch*	*Computation*	*Print*
Day	4,200	150	400
Night	9,200	250	650

Formulate the linear programming problem in order to determine the mixture of standard jobs that should be accepted during the day and night.

5. A firm places an order for a particular product at the beginning of each month and the product is received at the end of the month. The firm sells during the month from the stocks and it can sell any quantity. The prices at which the firm buys and sells vary every month. The following table shows the projected buying and selling prices for the next four months:

Months	*Selling price (₹) (during the month)*	*Purchase price (₹) (beginning of the month)*
April	—	75
May	90	75
June	60	60
July	75	—

The firm has no stocks on hand as on 01 April, and does not wish to have any stocks at the end of July. The firm has a warehouse of limited size which can hold a maximum of 150 units of the product.

The problem is to determine the number of units to buy and sell each month to maximise the profits from its operations. Formulate this problem as a linear programming problem.

6. The financial advisory board for company is reviewing new investment proposals for the coming year. Five projects have been identified as desirable. The net present value of the profits from each project, the cash requirements for each during the next 4 years and the amount of cash that will be available for investment in each of the 4 years are shown below:

	Cash needs (000's ₹) for year				*Net present value* of the projects over planning horizon *(000's ₹)*
Projects	1	2	3	4	
A	50	100	120	80	1250
B	200	125	70	40	800
C	100	100	100	100	1000
D	80	60	40	10	400
E	30	70	100	120	600
Cash available	200	300	250	200	

Formulate the problem as a LP problem, so as to maximise the total net present value of profits from the projects over the planning horizon.

7. To maintain his health, a person must fulfil certain minimum daily requirement for several kinds of nutrients. Assume that there are only three kinds of nutrients—calcium, protein and calories and the persons diet consists of only two food items, I and II, whose price and nutrient contents are shown in the table below.

	Food I (per lb)	*Food II (per lb)*	*Minimum daily* requirement for the *nutrient*
Price (₹)	0.60	1.00	—
Calcium	10	4	20
Protein	5	5	20
Calories	2	6	12

What combination of the two food items will satisfy the daily requirement and entail the least cost?

8. A steel plant produces scrap in mixed lots of three goods 1, 2 and 3. The plant requires atleast 90 tons of grade 1 scrap, 85 tons of grade 2 scrap and 75 tons of grade 3 scrap. It can produce any quantity from two sources A and B. Experience shows that sources give the following proposition of graded scraps.

Source	*Grade 1*	*Grade 2*	*Grade 3*
A	30%	40%	30%
B	40%	30%	30%

If the costs of source A is ₹ 500 per ton and source B is ₹ 400 per ton, determine how many tons of two should be mixed for fulfilling its scrap requirement at minimum cost.

9. A company manufactures two models of garden roller X and Y. When preparing the 1990 budget, it was found that the limitations on capacity were represented by the following weekly production maxima:

	Foundry	*Machine shop*	*Contribution per* model
Model X	160	200	₹ 120
Model Y	240	150	₹ 90

In addition, the material required for model X was in short supply and only sufficient for 140 units per week could be guaranteed for the year. Determine the optimal combination of output.

10. A new book has just been written and sent to the publisher. The ABC Printing company has obtained the contract for printing and binding the books. They have decided to produce the book with three types of binding: Paper back, Hard Cover and Deluxe Bound. The minimum quantity of each type to be bound are paper backs 5,500 copies; hard cover 3,700 copies and deluxe bound 2,500 copies. The company has two binding machines to process the books. The company has the maximum operating hours of 150 per machine 1 and 200 hours of machine 2. Binding machine 1 cost ₹ 12 per hour to operate and can produce either 50 paperbacks per hour, 40 hard cover per hour or 30 deluxe bound books per hour. Binding machine 2 costs ₹ 15 per hour to operate and can produce either 65 paper backs per hour, 35 hard covers per hour or 25 deluxe bound books per hour.

Formulate a linear programming model to determine how many hours each type of book should be processed on each machine, if Mr Anand, the operating manager, wishes to minimise production costs.

11. A pharmaceutical company has 100 kg of A, 180 kg of B and 120 kg of C available per month. They can use these materials to make three basic pharmaceutical products, namely 5-10-5, 5-5-10 and 20-5-10, where the numbers in each case represent the percentage by weight of A, B and C respectively in each of the products. The costs of these raw materials are given below:

Ingredients	*Cost per kg (₹)*
A	80
B	20
C	50
Inert ingredient	20

Selling prices of these products are ₹ 40.50, ₹ 43 and ₹ 45 per kg receptively. There is a capacity restriction of the company for the product 5-10-5, so that they cannot produce more than 30 kg per month. Determine how much of each of the products they produce in order to maximise their monthly profit.

12. A refinery makes 3 grades of petrol (A, B, C) from 3 crude oils (*d*, *e*, *f*). Crude *f* can be used in any grade, but others must satisfy the following specifications.

Grade	*Specifications*	*Selling price per litre*
A	Not less than 50% crude *d* Not more than 25% crude *e*	8.0
B	Not less than 25% crude *d* Not more than 50% crude *e*	6.5
C	No specifications	5.5

There are capacity limitations on the amounts of the three crude elements that can be used:

Crude	*Capacity*	*Price/litre*
d	500	9.5
e	500	9.5
f	300	6.5

It is required to produce the maximum profit. Formulate an LP model for the problem.

13. Consider the following problem faced by a production planner, in a soft drink plant. He has two bottling machines A and B. A is designed for 250 ml bottles and B for 500 ml bottles. However, each can be used in both types with some loss of efficiency. The following data is available:

Machine	*250 ml bottles*	*500 ml bottles*
A	100 bottles per minute	40 bottles per minute
B	60 bottles per minute	75 bottles per minute

The machine can be run 8 hours per day, 5 days per week. Profit on 250 ml bottle is 15 paise and on 500 ml bottle is 25 paise. Weekly production of the drink cannot exceed 9,000 litres and the market can absorb 25,000 of 250 ml bottles and 7,000 of 500 ml bottles per week. The planner wishes to maximise his profit, subject to all the production and marketing restrictions. Formulate this as a linear programme problem.

14. A car dealer selects his cars for sale very carefully so as to ensure optimisation of his profits. He deals in 4 types of cars A, B, F and G. The purchase value of the cars range at ₹ 60,000, 1,50,000, 55,000 and 2,20,000 and the sales value is fixed at ₹ 80,000, 1,75,00, 75,000 and 2,50,000 respectively. The probability of sale are 0.8, 0.9, 0.6 and 0.50 respectively during a period of six months. In order to invest ₹ 20,00,000 in his deals, he wishes to maintain the rates of purchase of cars as 3: 1: 2: 4. Work out how and how much he should buy. Formulate this problem as LP model.

15. A manufacturing unit makes three products using three processes i.e. casting, welding and machinery. The costs of casting are ₹ 350, 450 and 1050 per piece. The welding costs work out to ₹ 50, 70, 50 per piece and machining costs as ₹ 570, 650 and 950 per piece respectively. The selling prices of three products are fixed as ₹ 1,200, 1,350 and 2,700 respectively. Welding capacity available is only 25 per hour and machining capacity as 5 units per hour. The castings are purchased from an outsource and quantity is easily available on demand. Formulate the problem as LP model for maximisation of profit for the manufacturing unit.

16. Anita Electric company produces two products P_1 and P_2 that are produced and sold on a weekly basis. The weekly production cannot exceed 25 for product P_1 and 35 for product P_2 because of limited available facilities. The company employs a total of 60 workers.

Product P_1 requires 2 man-weeks of labour whereas P_2 requires only one. Profit margin on P_1 is ₹ 60 and on P_2 is ₹ 40. Formulate it as LPP and solve for maximum profit.

17. A local business firm is planning to advertise a special sale on radio and television during a particular week. A maximum budget of ₹ 16,000 is approved for this purpose. It is found that radio commericals cost ₹ 800 per 30 second spot with minimum contract of 5 spots. Television commercials, on the other hand, costs ₹ 4,000 per spot. Because of heavy demand, only 4 television spots are still available in the week. Also, it is believed that a TV spot is 6-times as effective as a radio spot in reaching consumers. How should the firm allocate its advertising budget to attract the largest number of consumers? How will the optimal solution be affected, if the availability of TV spots is not constrained?

18. A company manufactures two kinds of machines, each requiring a different manufacturing technique. The deluxe machine requires 18 hours of labour, 4 hours of testing and yields a profit of ₹ 400. The standard machine requires 3 hours of labour, 4 hours of testing and yields a profit of ₹ 200. There are 800 hours of labour and 600 hours of testing available each month. A marketing forecast has shown that the monthly demand for the standard machine to be no more than 150. Management wants to know the number of each model to produce monthly that will maximise total profit. Formulate the problem as LPP.

4.9 Summary

This unit is summarised by the following few points:

- **Constraints:** Restrictions identified for a business decision.
- **Decision Variables:** Factors responsible to affect the decision making.
- **Linear Programming:** A technique of choosing the best alternative from a set of available feasible options for decision making with linear constrants and linear objective function.
- **Objective Function:** A linear mathematic expression indicating the relationship of all variables associated with business problem.
- **Optimal Solution:** The best possible decision under the given circumstances.

4.10 Key Words/Abbreviations

- **Objective function:** The function that it is desired to maximize or minimize.
- **Decision variable:** Describe the quantities that the decision makers would like to determine
- **Constraints:** An inequality or equality defining limitations on decisions.
- **Opportunity loss:** It is defined as the difference between the optimal payoff and the actual payoff received.
- **Iterative method:** It means repeatedly carrying out a process.

4.11 Learning Activity

1. Why do we call the above technique as Linear Programming?

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2. In the technique of Linear Programming really very useful to the business managers for their day-to-day functions? If so, in what all areas?

   ----

3. Can we use this technique of Linear Programming for manpower planning as well as for advertising function of business marketing? If so, how?

4.12 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. What are the assumption underlying linear programming?
2. Explain advantages of linear programming?
3. Describe the steps for formulation of linear programming?

B. Multiple Choice/Objective Type Questions

1. The useful technique to obtain optimum use of productive resources is ____________.

    (a) Linear programming (b) Transportation problem

    (c) CPM (d) PERT

2. A linear mathematic expression indicating the relationship of all variables associated with business problem is known as _________________.

    (a) Constraints (b) Decision Variables

    (c) Optimal Solution (d) Objective Function

3. A set of feasible solutions of the system in the form of a convex polyhedron, where any two points of polygon, selected arbitrarily and joined to gather by straight line lies completely within the polygon is called as ______________.

    (a) Convex Polygon (b) Convex set

    (c) Concave polygon (d) None of these

4. Which of the following are assumption under linear programming?

    (a) Proportionality (b) Additivity

    (c) Certainty (d) All of these

5. The constraints of Maximisation problem are of ____________________.

   (a) Greater than or equal type (b) Less than or equal type

   (c) Less than type (d) Greater than type

6. The entries whose values are to be determined from the solution of the LPP is called __________________.

   (a) Decision variable (b) Constraints

   (c) Opportunity loss (d) Objective function

7. In LPP, Graphical method can be used for______________________.

   (a) Four variable problem (b) Three variable problem

   (c) Two variable problem (d) None of these

8. In LPP, Greater than or equal to constraints the feasible region of the solution lies ________

   (a) Inside the constraint line (b) Outside the constraint line

   (c) On the line (d) None of these

9. The Constraint which does not affect the optimal solution is called as ________________.

   (a) Redundant Constraint (b) Mixed Constraint

   (c) Infinite Constraint (d) None of these

10. The line passes through the point of optimal solution on the feasible region is ____________.

    (a) Iso-profit line (b) Iso-cost line

    (c) Iso -profit or iso- cost line (d) None of these

11. Simplex method is an ________________________.

    (a) Iterative method (b) Decision theory

    (c) Inventory models (d) None of these

Answers:

1. (a), 2. (d), 3. (b), 4. (d), 5. (b), 6. (d), 7. (c), 8. (b), 9. (a), 10. (c), 11. (a).

4.13 References

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UNIT 5 SOLVING LPP – THE SIMPLEX METHOD 1

Structure:

5.0 Learning Objectives

After studying this unit, you will be able to:

- Explain solution of LP problems through graphical method.
- Discuss concept of infeasible solution by graphical means.
- Differentiate the concept of extreme point approach and Convex Polygon.
- Define the concept of Iso-profit (cost) Function Approach.
- Illustrate through various solved examples.
- Analyse Yourself through self-assessment problems.
- Solve LPP using simplex method
- Illustrate flow chart of simplex method

5.1 Introduction

When a large number of variables (more than 2) are involved in a problem, the solution by graphical method is not possible. The simplex method provides an efficient technique which can be applied for solving LPPs of any magnitude, involving two or more decision variables. In this method, the objective function is used to control the development and evaluation of each feasible solution of the problem.

The simplex Algorithm is an iterative procedure for finding, in a systematic manner, the optimal solution that comes from the corner points of the feasible region. Simplex algorithm considers only those feasible solutions which are provided by the corner points and that too not all of them. It is very efficient algorithm. The technique also has the merit to indicate whether a given solution is optimal or not. This method was formulated by G.B. Dantzig in 1947.

5.2 Method of Application of Simplex Method

1. First of all, an appropriately selected set of variables are introduced into the problem.
2. Iterative process is started by assigning values only to those variables so introduced and the primary decision variables of the problem are all set equal to zero.
3. The algorithm then replaces one of the initial variables by another variable—the variable which contributes most to the desired optimal value enters in, while the variable creating the bottleneck to the optimal solution goes out. This improves the value of the objective function.
4. The procedure of substitution of variables is repeated until no further improvement in the objective function value is possible. The algorithm terminates thus indicating that the optimal solution has been obtained.

   For application of simplex method, following conditions must be satisfied.

   (*a*) Right Hand Side (RHS) of each constraint should be non-negative. In case of negative RHS, the whole solution (inequality) to be multiplied by -1.

   (*b*) Each of the decision variables of the problem should be non-negative. In case of ‘unrestricted' variables it is treated as the difference of two non-negative variables- such as x_1, $x_2 \geq 0$, x_3 unrestricted can be written as x_1, x_2, x_4, $x_5 \geq 0$, where $x_3 = x_4 - x_5$. After the solution is reached, we substitute difference of x_4 and x_5 as x_3.

5.3 Extreme Point Approach and Convex Polygon

This solution by Graphical method for an LP problem can be divided into five successive steps.

Step 1: Formulate the problem into LP as described in Unit 4.

Step 2: Graph the limitations or constraints, initially ignoring the inequalities and decide the are of feasible region, taking into account the inequality of the relationships. This feasible region should be indicated in the form of a convex polygon.

Step 3: Determine the point locations of the extreme points of the feasible region.

Step 4: Evaluate the value of the objective function at all these extreme points.

Step 5: Determine the extreme point to obtain the best or optimal value become the value of decision variables from where the value of the function becomes optimal.

The cases can best be demonstrated by analysing maximisation and minimisation problems.

5.4 Clarification on Application of Various Steps

We have used the procedure as given below:

Step 1: LP model formulation has been dispensed with by selecting the mathematical model as given in all these problems.

Step 2: In all the three cases, the inequalities have been converted into equalities to obtain graphical form of the constraints. Constraint $2x_1 + 3x_2 = 60$ has been drawn into a straight line as $2x_1 + 3x_2 = 60$ by using $x_1 = 0$ to get one extreme end of the line as $x_2 = 20$ and by putting $x_2 = 0$, getting the other extremity of the line as $x_1 = 30$. Thus line PT represents the Material constraints so marked on the graph.

After having drawn the other line as SR for labour constraints in the similar way, we obtain the feasible region by using inequality conditions as given. Since the labour constraint is $2x_1 + 3x_2 \leq 60$, and the line drawn is for $2x_1 + 3x_2 = 60$, the region has to be below this line, so is the case for labour constraint and hence feasible region so obtained will be bounded by OPQR and it is a convex polygon.

Step 3: The point locations of the extreme points of the feasible region have since been determined as O, P, Q and R, whose co-ordinates are as indicated, while working out the valves of Z for all these points.

Step 4: The evaluation of the obtective function as Z = $40x_2 + 35x_2$ is done for all points for feasible region extremities. This is indicated in the table so drawn.

Step 5: Having obtained the values of Z for all the extreme points, we select the maximum value of the objective function as 1,000, as desired in the problem. The location of such a point indicating maximum value of the objective function, thus, becomes the values of the decision variables. In this case under discussion, it comes out to be $x_1 = 18$ and $x_2 = 8$ to given maximum profit of 1,000 as indicated. Other problems also can be understood in the similar way under various steps mentioned above.

5.5 ISO-Profit (Cost) Function Approach

This approach is at slight variance from the earlier method of corner points described above. The major steps of this approach are as under:

Step 1: Identify the feasible region and extreme points of this region.

Step 2: Draw an iso-profit or iso-cost line for a particular value of the objective function. As the name implies, the cost/profit on all points is the same.

Step 3: Move the iso-cost/profit lines parallel in the direction of increasing/decreasing objective function values.

Step 4: The feasible extreme point is then located, for which the solution is optimal i.e., where iso-profit/iso-cost is largest/smallest.

5.6 Solved Examples

Problem 1:

A and B are two products to be manufactured. Unit profits are ₹ 40 and ₹ 35 respectively. Max. material available is 60 kgs and labour 96 hours. Each unit of A needs 2 kg of material and 3 manhours, whereas each unit of B needs 4 kg of material and 3 manhours. Find optimal level of A and B to be manufactured.

Solution:

$$\begin{array}{llcll} & \text{Maximise } Z & = & 40x_1 + 35x_2 & \text{Profit} \\ \text{Subject to,} & 2x_1 + 3x_2 & \leq & 60 & \text{Material constraint} \\ & 4x_1 + 3x_2 & \leq & 96 & \text{Labour constraint} \end{array}$$

$x_1, x_2 \geq 0$ Positive values of decision variables

where x_1 and x_2 are the quantities of A and B to be manufactured and sold.

Marking the constraint restrictions on the chart, we get a graphical representation as follows (Fig. 5.1)

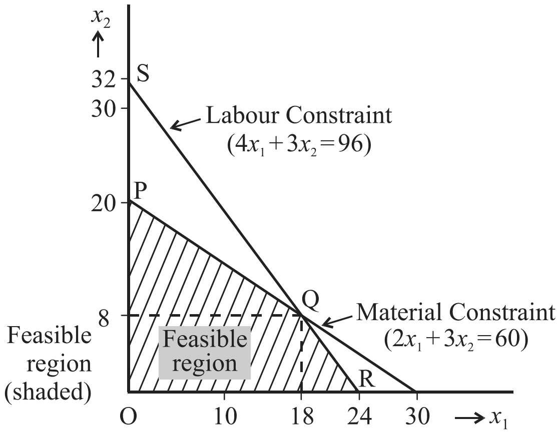

Fig. 5.1

These lines are obtained by putting extreme values of x_1 and x_2 ignoring the inequalities. Thus Material constraint and Labour constraint will be indicated by lines shown on the graph.

In the above example, the feasible region is formed by a four sided polygon *OPQR*. It may be observed that although the feasible region is determined by the constraints of the given system, this region must constitute a CONVEX SET, and if it does not, linear programming cannot be applied. The concept of convex set in the context of a two variable problem can be understood as follows– if any two points are selected in the region and the line segment formed by joining these two points lies completely in this region, including on its boundary, then this region represents a convex set. Thus the feasible region of the figure above is a convex set.

Obtaining the Optimal Solution: Although all points in the feasible region represent feasible decision alternatives, they are not all equally important. Optimal point should lie at one of the corners or extreme points of the feasible region polygon. The co-ordinates of each of these points should be obtained by solving two constraint inequalities converted to equations.

This value can be read directly from the graph as the point of their intersection at *Q*.

Where from $x_1 = 18$, $x_2 = 8$.

Values of *Z* corresponding to all corner points are:

Point	x_1	x_2	*Z*	
O	0	0	0	
P	0	20	700	
Q	18	8	1,000	(max.)
R	24	0	960	

Since value of *Z* is maximum at *Q*, optimal solution is to produce 18 units of product A and 8 units of product B every week so as to get the profit of ₹ 1,000. By this combination, the material as well as labour resources are fully utilised.

Problem 2:

Let us consider the following problem

	Minimise Z	$=$	$40x_1 + 24x_2$	Total cost
Subject to,	$20x_1 + 50x_2$	$\geq$	$4{,}800$	Material I requirement
	$80x_1 + 50x_2$	$\geq$	$7{,}200$	Material II requirement
	x_1, x_2	$\geq$	$0.$	Non-negative condition.

Solution:

Graphically, this can be represented as given below, (Fig. 5.2).

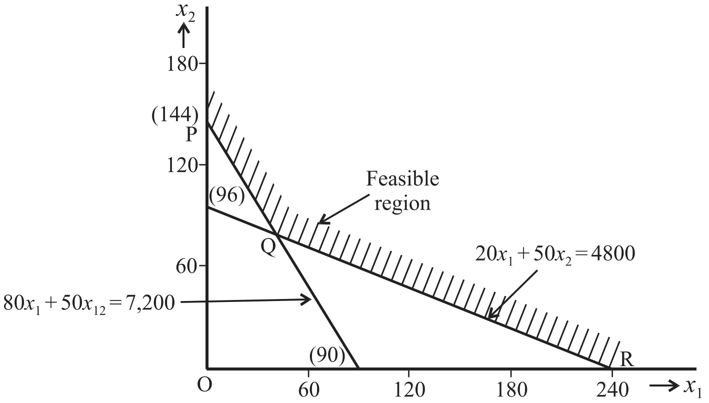

Fig. 5.2

The feasible region represents a convex set. However, it is not bounded from all the sides as none of the restrictions place an upper limit on the value of either of the decision variables. In order to obtain the optimal solution, the co-ordinates of *P*, *Q*, *R* are located and value of objective function evaluated as follows:

Point	x_1	x_2	Z	
P	0	144	3,456	(minimum)
Q	40	80	3,520	
R	240	0	9,600	

Since the total cost is minimum at point *P*, the optimal solution to the problem is to buy 144 units of product B (x_2) and none of product A (x_1). This would entail a total cost of ₹ 3,456. The requirements for both the materials *i.e.,* material I and II can be worked out for 144 units of product B. *i.e.,* material I as 7,200 units and material II also as 7,200 units.

Problem 3:

$$\text{Maximise } Z = 2x_1 + 5x_2$$

Subject to,

$$x_1 \leq 4$$

$$x_2 \leq 3$$

$$x_1 + 2x_2 \leq 8$$

$$x_1;\ x_2 \geq 0$$

Solution:

The plotting of the above relationships is obtained as follows: (Fig. 5.3)

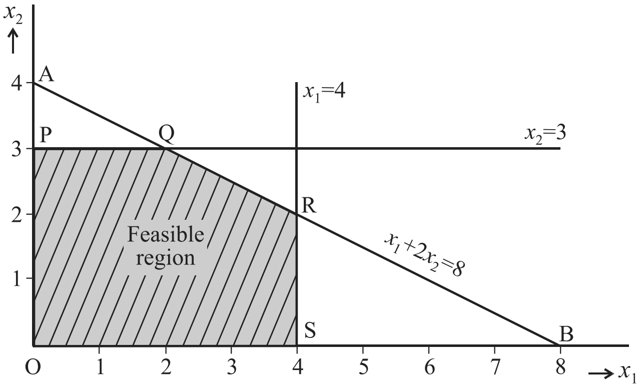

Fig. 5.3

From the inequalities relationship, feasible region for solution has been obtained and so marked as *OPQRS*. By taking values of points *O, P*, *Q*, *R* and *S*, we can calculate the values of *Z*.

Point	x_1	x_2	*Z*	
O	0	0	0	
P	0	3	15	
Q	2	3	19	(max.)
R	4	2	18	
S	4	0	8	

Maximum *Z* can be established at point *Q* and the value of *Z*(max.) = 19

Thus the product mix for maximum *Z* (say profit) will be 2 units of x_1 and 3 units of x_2.

Problem 4:

A firm makes two products X and Y and has a total production capacity of 9 tonnes per day, X and Y requiring the same production capacity. The firm has a permanent contract to supply atleast 2 tonnes of X and atleast 3 tonnes of Y per day to another company. Each tonne of X requires 20 machine hours production time and each tonne of Y requires 50 machine hours production time; the daily maximum possible number of machine hours is 360. All the firm's output can be sold and the profit made is ₹ 80 per tonne of X and ₹ 120 per tonne of Y. It is required to determine the production schedule for maximum profit and to calculate this profit.

Solution:

***Step 1*:** Formulating the problem into an LP model *i.e.,* mathematical form, we can write as follows:

Maximise Profit (Z) = $80x_1 + 120x_2$

where x_1 = quantity of X produce in tonnes

x_2 = quantity of Y product in tonnes

The constraints can be formulated as follows:

$x_1 + x_2$	$\leq$	9	Production capacity
x_1	$\geq$	2	Supply constraint
x_2	$\geq$	3	Supply constraint
$20x_1 + 50x_2$	$\leq$	360	Machine-hours constraint
and x_1, x_2	$\geq$	0	Non-negative constraint

These constraints have been drawn on the graph on the previous page.

Step 2. Treating inequalities as equalities, we draw straight lines as given in the Fig. 5.4 indicating the feasible region also (shaded) based on inequality provisions.

The feasible region is as marked (shaded) bounded by *PQRS*.

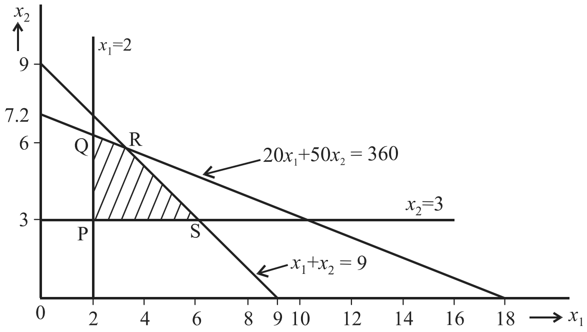

Fig. 5.4

Step 3. The extreme points of the feasible region are $P(2,3)$, $Q(2,6.4)$, $R(3,6)$ and $S(6,3)$.

Step 4. The values of *Z* (Profit) for all these extreme points are given below:

Point	*Co-ordinates of Corner points*	*Objective function*
P	(2, 3)	$Z_P = 520$
Q	(2, 6.4)	$Z_Q = 928$
R	(3, 6)	$Z_R = 960$ (max)
S	(6, 3)	$Z_S = 840$

Step 5. The maximum value of the objective function *Z* comes out as 960 and hence the values of the decision variables (optimal) will be

x_1 = 3 (tonnes of X to be produced)

x_2 = 6 (tonnes of Y to be produced)

and Maximum Profit = ₹ **960**

Problem 5:

An animal feed company must produce 200 kg of a mixture consisting of ingredients x_1 and x_2. The ingredient x_1 costs ₹ 3 per kg and x_2 cost ₹ 5 per kg. Not more than 80 kg of x_1 can be used and atleast 60 kg of x_2 must be used. Find the minimum cost mixture.

Solution:

The mathematical LP model for the given problem is as under

Minimise (total cost) = $3x_1 + 5x_2$

Subject to, $x_1 + x_2 = 200$

$x_1 \leq 80$

$x_2 \geq 60$

and $x_1, x_2 \geq 0.$

Drawing these lines for the constraints, we get the feasible region as follows (Fig. 5.5)

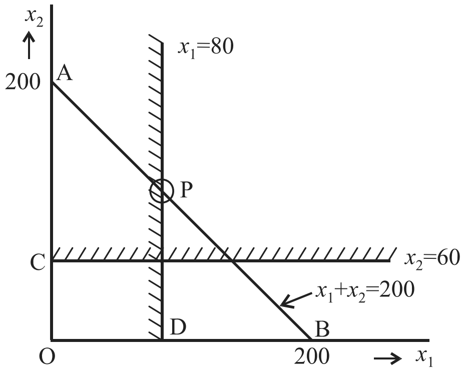

Fig. 5.5

It can be seen that three is no feasible region in this case due to the constraint but there is only a feasible point (P), whose co-ordinates are $x_1 = 80$, $x_2 = 120$.

$$\text{Cost} = 3 \times 80 + 5 \times 120$$

$$= ₹\ 840.$$

Hence, the optimal mix of the animal feed is

80 kg of produce X_1

120 kg of produce X_2

at Minimum cost = **₹ 840.**

Problem 6:

Use the graphical method to solve the following LP problem.

$$\text{Maximise } Z = 5x_1 + 2x_2$$

subject to,

$$2x_1 + 3x_2 \leq 150$$

$$3x_1 \leq 150$$

$$5x_2 \leq 200$$

and

$$x_1;\ x_2 \geq 0.$$

Solution:

Since the LP problem is available in the mathematical form, we can proceed to draw the graph for the constraints as follows:

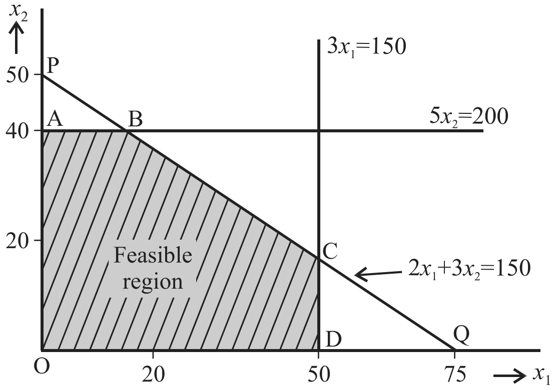

Fig. 5.6

The feasible region is bounded by extreme corners as A, B, C, D and O. The co-ordinates of these points and the values of objective function for the same are as under (Refer Fig. 5.6).

Extreme points	*Co-ordinates*	*Value of Z*	
A	(0, 40)	80	
B	(15, 40)	155	
C	(50, 17)	17	(max.)
D	(50, 0)	250	
O	(0, 0)	0	

Here the maximum value of the objective function is at point C, whose co-ordinates can be written as (50, 17).

Hence $x_1 = 50$

$x_2 = 17$

Max $Z = 284$

Problem 7:

Solve the LP problem given below, using graphical method.

$$\text{Maximise } Z = 2x_1 + x_2$$

subject to,

$$x_1 + 2x_2 \leq 10$$

$$x_1 + x_2 \leq 8$$

$$x_1 - x_2 \leq 2$$

$$x_1 - 2x_2 \leq 2$$

$$x_1, x_2 \geq 0$$

Solution:

Since the problem is given in the mathematical form, we can proceed with drawing the graph by using inequalities as equalities. The graph so obtained is given below. (Fig. 5.7)

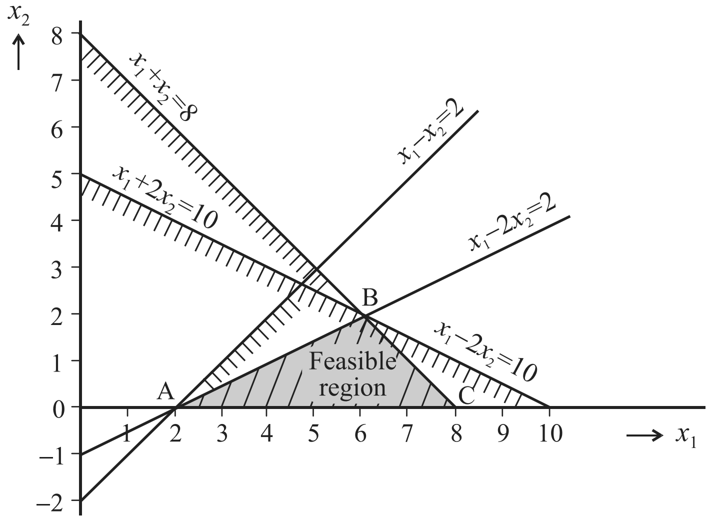

Fig. 5.7

The feasible region given above as shaded space has its corner points as A, B, C whose co-ordinates and corresponding values of objective functions are as follows:

Corner Point	*Co-ordinates*	*Values of Z*	
A	(0, 2)	2	
B	(6, 2)	14	(max.)
C	(0, 8)	8	

The maximum value of the objective function being 14 at point B (6, 2), we get the solution as

$$x_1 = 6, \quad x_2 = 2$$

and $$\text{Max. } (Z) = 14$$

Problem 8:

Solve the following LP problem by graphical method:

$$\text{Minimise } Z = -x_1 + 2x_2$$

subject to,
$$-x_1 + 3x_2 \leq 10$$
$$x_1 + x_2 \leq 6$$
$$x_1 - x_2 \leq 2$$

and $$x_1, x_2 \geq 0$$

Solution:

Since the problem has been stated in the mathematical form, we can proceed to draw the graphical representation of all the constraints. (Refer Fig. 5.8)

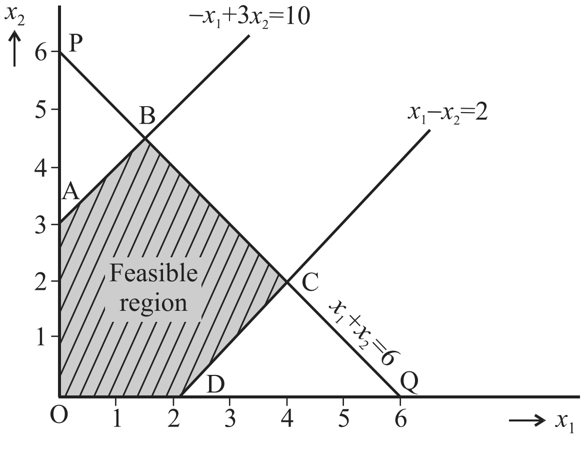

Fig. 5.8

From the graph and location of feasible region represented by the polygon A, B, C, D, O due to the nature of constraints, we locate the extreme points of the region as follows:

Extreme Points	*Co-ordinates*	*Values of Objective Function*	
A	(0, 3)	6	
B	(1.5, 4.5)	10.5	
C	(4, 2)	0	
D	(2, 0)	–2	(minimum)
O	(0, 0)	0	

From the above tabulation, we can infer that the value of objective function is minimum at point D, which is (2, 0) and hence optimal solution is,

$$x_1 = 2,$$

$$x_2 = 0$$

and $$\text{Min } Z = -2$$

Problem 9:

Use the graphical method to solve the following LP problem:

$$\text{Minimise } Z = 20x_1 + 10x_2$$

subject to,

$$x_1 + 2x_2 \leq 40$$

$$3x_1 + x_2 \geq 30$$

$$4x_1 + 3x_2 \geq 60$$

$$x_1, x_2 \geq 0$$

Solution:

Since the problem is stated in the mathematical form, we can proceed to draw the graph relating to the constraints by using equations rather than the inequalities of the constraints. The graphical representation is given in Fig. 5.9.

From the graph, the feasible region can be seen as represented by points A, B, C and D. The co-ordinates of the corner points of the feasible region are A (6, 12), B (4, 18), C (40, 0) and D (15, 0). The evaluation of the objective function is given below.

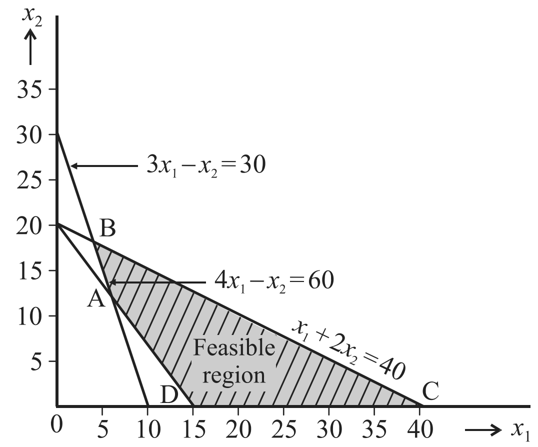

Fig. 5.9

Corner Points	*Co-ordinates*	*Values of objective function (Z)*	
A	(6, 12)	240	(min.)
B	(4, 18)	260	
C	(40, 0)	800	
D	(15, 0)	300	

The minimum value of the objective function being 240 occuring at point A (6, 12), we get

$$x_1 = 6$$

$$x_2 = 12$$

and minimum $Z = 240$

Problem 10:

Let us consider the following LP problem for optimal solution by using Iso-Profit approach

$$\text{Maximise } Z = 3x_1 + x_2$$

subject to,

$$x_1 + 2x_2 \leq 20$$

$$2x_1 + x_2 \leq 10$$

$$x_1 \leq 5$$

$$x_2 \leq 7$$

$$x_1, x_2 \geq 0$$

Solution:

Let us first draw the graphical representation of the above problem as given: (Fig. 5.10)

Iso profit objective function line can be drawn as follows:

Since $Z = 3x_1 + x_2$

$\therefore$ $X_2 = Z - 3x_1$

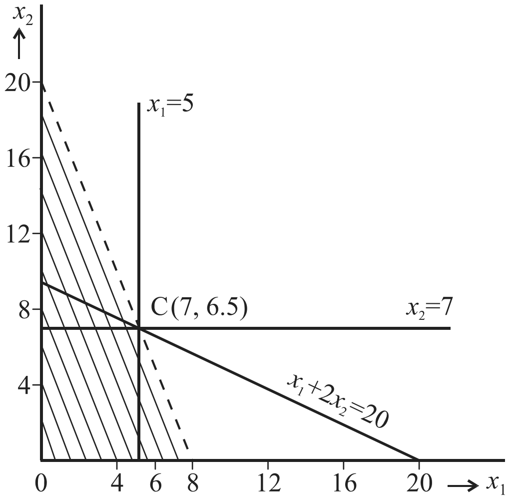

Fig. 5.10

Since Z cannot be determined, the line can be drawn based on slope (–3). We go on increasing the intercept on x_2 at 3, 6, 9, 12, 15 etc. and on x_1 axis as 1, 2, 3, 4, 5, etc. The value of the objective function will be at the point when Iso-profit line passes the point C. Max. $Z = 27.5$ at point (7, 6.5)

Hence, $x_1 = 7$, $x_2 = 6.5$ and Max. $Z = 27.5$.

Problem 11:

Use the graphical method to solve the following LP problem

$$\text{Maximise } Z = 3x_1 + 2x_2$$

subject to,

$$x_1 + x_2 \leq 3$$

$$x_1 - x_2 \geq 2$$

$$x_1 \leq 1$$

$$x_1, x_2 \geq 0$$

Solution:

Following normal system of drawing constraints on the graph, we get the following Fig. 5.11.

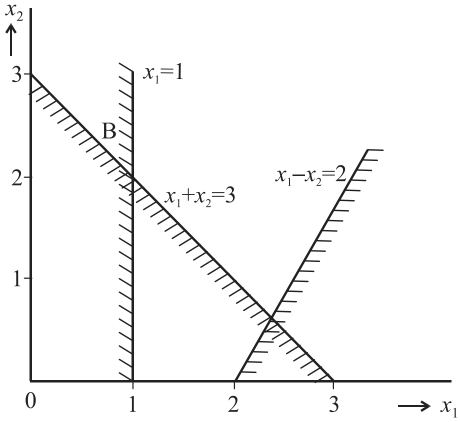

Fig. 5.11

This indicates the feasible region to be unbounded and hence the problem has an unbounded solution.

Problem 12:

Consider the LP problem given by following statement to be solved by using graphical method, for optimal solution

$$\text{Maximise } Z = 40x_1 + 60x_2$$

subject to,

$$2x_1 + x_2 \geq 70$$

$$x_1 + x_2 \leq 40$$

$$x_1 + 3x_2 \geq 90$$

and

$$x_1, x_2 \geq 0$$

Solution:

The mathematical problem can be solved by drawing the graph of the constraint in the prescribed manner. (Refer Fig. 5.12 below)

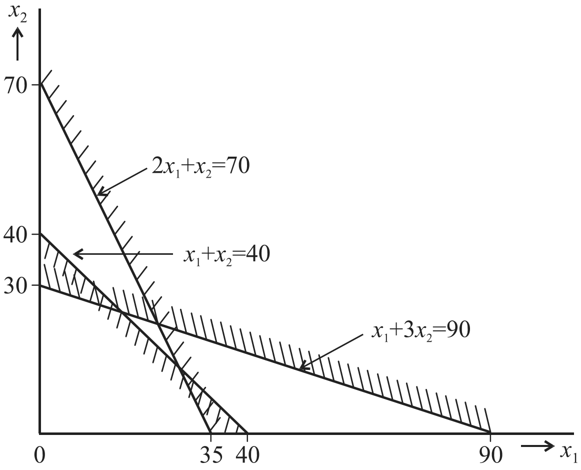

Fig. 5.12

The feasible region can not be obtained as all the constraints are not satisfied simultaneously. Hence, the solution is infeasible.

Problem 13:

Use graphical method to solve the following LP problem

$$\text{Maximise } Z = 6x_1 + 4x_2$$

$$\begin{aligned} \text{subject to,} \quad & 2x_1 + 4x_2 \leq 4 \\ & 4x_1 + 8x_2 \geq 16 \\ \text{and} \quad & x_1\ x_2 \geq 0 \end{aligned}$$

Solution:

Let us draw the graph for the constraints for the given mathematical form of LP problem. (Refer Fig. 5.13).

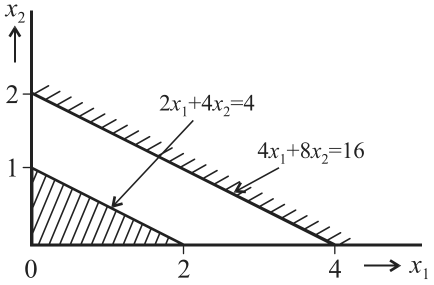

Fig. 5.13

It can thus be seen that the inequalities given in the problem are inconsistent and that there is no set of points satisfying all the constraints. Hence, the problem does not have a feasible solution.

5.7 Simplex Algorithm (Maximisation Case)

The steps involved in the Simplex Algorithm in order to obtain an optimal solution from a standard linear programming problem are as under:

Step 1:

(*a*) Formulation of the LP (mathematical) model—The method of formulation of an LP model has been amply narrated and explained in Chapter 16.

(*b*) The standard form of the simplex model is obtained by ensuring all inequalities are of the type £, if not, by multiplying the relevant constraint by -1 on both sides.

(*c*) Similarly, if the problem is that of minimisation, it need to be converted into standard form by subtracting slack variable and adding artificial variables.

(*d*) Addition of variable to the resources inequalities will ensure their conversion into equalities. Also we assign a zero-cost coefficient of these added variables in the objective function.

Step 2: *Set up Initial Solution*: The initial solution of the simplex method is obtained by assigning all decision variable coefficients as zero in order to begin the solution from zero level i.e., the value of the objective function as zero and then gradually improving it by iteration process. The simplex table contains a row for *cj, Ƶj*, the variable row (decision, slack as well as artificial variables as applicable), the initial variable mix column without decision variables, respective cost coefficients of each variable in the row just above the corresponding variable row, the values of resource quantities (*bi*'s) and then working out the D*j* = *Cj* – *Ƶj* or *Ƶj* – *Cj* (can be done both ways). *Ƶj* are calculated by multiplying their unit cost element *i.e. Ƶj* (unit) column with the quantities of corresponding variables in their respective column.

Step 3: *Testing the Solution for Optimality*: On the careful examination of the Index row, if all the elements of *Cj* - *Ƶj* row for a maximisation problem are negative or zero, it is optimal solution. In case the Index row is obtained as *Ƶj* - *Cj*, then all elements of the row have to be positive or zero, to make the solution optimal. If the solution is not optimal, it can be improved by next iteration by replacing one of the basic variables by the entering variable as decided by step 4.

Step 4:

(*a*) *Identify Entering Variable*: (*i*) In case of D*j* = *Cj* – *Ƶj*, the improvement can be brought about by selecting the column with highest positive value of the D*j*. Thus corresponding variable becomes the candidate as entering variable. This is called the key column. If the row of D*j* has been obtained as D*j* = *Ƶj* – *Cj*, then key column indicating Entering Variable corresponds to the minimum value of the index row.

(*b*) *Identify Outgoing Variable*: Having identified the key column, all values of *bi*'s are divided by the corresponding elements of key column for each row to obtain the ratios bi/aij where aij are the corresponding values of elements of key column with the relevant row *j*. The outgoing variable is decided by the least positive ratio. The basis variable pertaining to this value is the outgoing variable *i.e.,* variable corresponding to the key row.

Step 5: *Identify Key Element*: The value at the point of intersection of key column and key row *i.e.* the entering and outgoing variable, is the key element.

Step 6: *Iteration for improved solution:*

(*a*) Replace outgoing variable with the entering variable and enter relevant coefficients in *Zj* column.

(*b*) Compute the key row with reference to the newly entered variable by dividing the old row quantities by the key element.

(*c*) Modify new values for the other rows. In the revised simplex table, all the other rows are recalculated as follows.

New row elements = Elements in the old row - [corresponding key column element multiplied by the corresponding new element of the revised row at (*b*) above.]

(*d*) The same procedure is followed for modification to *bi* column also.

(*e*) Having obtained the revised simplex table, evaluate D*j* = *Cj* - *Zj* and test for optimality as per Step 3 above.

Step *7.*

(*a*) *Obtaining Optimal Solution*—If the table indicates optimality level by examining D*j* or index row, the iteration stops at this point and values of *bi*'s for corresponding variables in the product mix column will indicate the values of the variables contributing towards the objective function. The value of the objective function can be then worked out by substituting these values for corresponding decision variables.

(*b*) If the solution is not optimal, proceed to Step 8.

Step 8. *Revise or Improve the Solution*—For this purpose, we repeat Step 4 to Step 7 till optimality conditions are fulfilled and solution is obtained as per Step 7.

Rule for Ties. Whenever two similar values are encountered in index row or ratio column, we select any column or ratio, but to reduce computation effort, following can be helpful.

(*a*) For key column, select the left most tie element.

(*b*) For ratio, select nearest to the top.

5.8 Flow Chart of Simplex Algorithm for Maximisation Cases

Flow Chart—Simplex Method (Maximisation Problem)

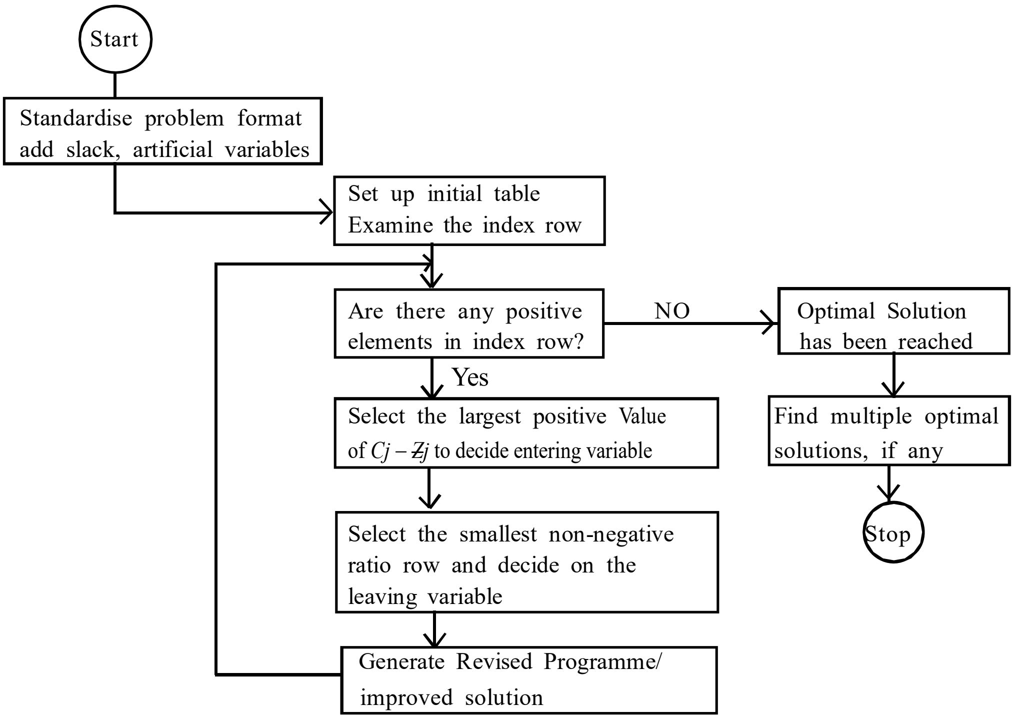

Fig. 5.14

Simplex algorithm is illustrated by solving an LP problem.

5.9 Summary

- **Feasible solution:** A Solution of the problem satisfying all the applicable constraints.
- **Convex polygon:** The polygon obtained by joining all possible feasible points, where any extreme points joined through straight lines.

- **Convex set:** A set of feasible solutions of the system in the form of a convex polyhedron, where any two points of polygon, selected arbitrarily and joined to gather by straight line lies completely within the polygon.
- **ISO-profit function:** A set of linear functions denoting the same profit level for a given objective function.

5.10 Key Words/Abbreviations

- **Infeasible Solution:** A linear program is infeasible if there exists no solution that satisfies all of the constraints
- **Unbounded Solution:** An unbounded solution of a linear programming problem is a situation where objective function is infinite
- **Shadow Price:** In linear programming problems the shadow price of a constraint is the difference between the optimised value of the objective function and the value of the objective function, evaluated at the optional basis, when the right hand side (RHS) of a constraint is increased by one unit.
- **Scarce Resource:** The scarce resources are the times available on the machines and the alternative activities are the individual production volumes.

5.11 Learning Activity

1. In the technique of Linear Programming really very useful to the business managers for their day-to-day functions ? If so, in what all areas?

   ----

2. Can we use this technique of Linear Programming for manpower planning as well as for advertising function of business marketing? If so, how?

5.12 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. What are the steps for simplex algorithm in order to obtain an optimal solution.
2. Explain optimal solution.
3. Solve the following problems graphically

(a) Max. $Z = 5x_1 + 3x_2$

subject to, $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

(b) Min. $Z = 4x_1 - 2x_2$

subject to, $x_1 + x_2 \leq 14$

$3x_1 + 2x_2 \geq 36$

$2x_1 + x_2 \leq 24$

$x_1, x_2 \geq 0$

4. The weight of a special purpose brick is 5 kg and it contains two basic ingredients B_1 and B_2. B_1 costs ₹ 5 per kg and B_2 costs ₹ 8 per kg. Strength consideration dictates that the brick contains not more than 4 kg of B_1 and a minimum of 2 kg of B_2. Since the demand for the production is likely to be related to the price of the brick, find out graphically, the minimum cost of the brick satisfying the above condition.

B. Multiple Choice/Objective Type Questions

1. The resources which are completely consumed in the production process is called ________.

(a) Scare resources (b) Abundant resources

(c) Artificial resources (d) None of these

2. If there is some slack quantity which is equal to the unutilized capacity is known as________.

(a) Scare resources (b) Abundant resources

(c) Artificial resources (d) None of these

3. The value in the simplex table which is at the intersection of key column and key row is called

(a) Key row (b) Key column

(c) Key element (d) None of these

4. The value of one extra unit of the resource is means that _______________.
   (a) Scarce resource (b) Abundant resource
   (c) Shadow price of a resource (d) None of these
5. When there is an incoming variable but there is no outgoing variable is called ____________.
   (a) unbounded solution (b) Error in problem
   (c) infeasible solution (d) None of these

Answers:

1. (a), 2. (b), 3. (c), 4. (c), 5. (a).

5.13 References

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UNIT 6 THE SIMPLEX METHOD 2

Structure:

6.0 Learning Objectives

After studying this unit, you will be able to:

- Explain the method of application of Simplex Method.
- Discuss the standard from of LP problem for use in Simplex Method.
- Differentiate the concept of slack and artificial variables for simplex solution.
- Define the basic definition of various terms used.
- Discuss the simplex algorithm for maximisation cases and modification for minimisation cases.
- Illustrate the flow chart of simplex Method for computer application.
- Explain the two methods for solving minimisation cases.
- Distinguish the concept of Degeneracy and Sensitivity analysis.
- Describe the use of simplex Method for business problems.
- Solve LPP using two phase simplex method
- Define degeneracy
- Explain Big-M method to LPP

6.1 Introduction

The simplex method provides an algorithm (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the fundamental theorem of linear programming. The simplex method is used to eradicate the issues in linear programming. It examines the feasible set's adjacent vertices in sequence to ensure that, at every new vertex, the objective function increases or is unaffected. The Simplex Method or Simplex Algorithm is used for calculating the optimal solution to the linear programming problem. In other words, the simplex algorithm is an iterative procedure carried systematically to determine the optimal solution from the set of feasible solutions.

6.2 Basic Definitions in Simplex Method

Before using the simplex method, let us examine and understand certain basic terms involved in the procedure.

1. **Standard Form:** This has already been clarified in the initial part of this chapter. With linear relationships of objective function and constraints, making RHS of constraints as equal produces standard form, whereas the inequality situation is called canonical form.
2. **Slack and Artificial Variables:** These have also been explained under an appropriate heading. Their physical significance have also been clarified. These are generally designated as S_1, S_2.....etc. and A_1, A_2 etc. respectively. Whereas the slack variables indicate spare capacity of the constraints, artificial variables are imaginary variables added for standard form.
3. **Surplus Variable:** A variable subtracted from the left hand side of a greater than or equal to constraint to convert the constraint into an equality. Physical sense or interpretation of the surplus variable is that it is amount of resource over and above the minimum required level. In case the constraint inequality is of the type "less than or equal to", then it is called slack variable.
4. **Basic Solution:** There may be n variables and m constraints in a linear programming problem. When we evaluate the solution of this problem by setting $(n - m)$ of the variables to zero and solve the other m variable equations, we obtain a unique solution. It is called "Basic Solution".
5. **Basic Feasible Solution:** When a basic solution satisfies even the non-negativity requirement, it is called Basic Feasible Solution. Since it has to be within the feasible region (graphical method), a basic feasible solution corresponds to a corner point of the feasible region.
6. **Simplex Table:** A table used for calculations during various iterations of the simplex procedure, is called Simplex Table.

7. **Variable Mix:** The values of the column that contains all the variables in the solution.
8. **Basis:** The set of variables which are not set to zero and figure in the column of "Product Mix" are said to be in the 'Basis'. Other than these figuring in the product mix column are termed as non-basic variables.
9. **Iteration:** Since the simplex procedure is that of constant improvement type from one basic feasible solution to another, these steps of moving from one solution to another to reach optimal solution are called Iterations.
10. **Cj Row:** It is the row containing the coefficients of all the variables (decision variables, slack or artificial variables) in the objective function.
11. **Constraints:** Restrictions on the problem solution arising from limited resources.
12. **C*j* - Z*j* = D*j* or Index Row:** The row containing net profit or loss resulting from introducing one unit of the variable in that column in the solution. A positive number in the D*j* row would indicate an algebraic reduction or increment in the objective function if one unit of the variable of that column is introduced in the basis.
13. **Key Column:** The column with the largest positive number in *Cj* - *Zj* row in a maximisation problem or the smallest number in a minimisation problem is called key column. This indicates the variable entering the solution in the next iteration by replacing an appropriate variable.
14. **Key Row:** When we work out the ratio of quantities *bi*'s and the elements of the key column, we get the last column of the simplex table. The outgoing variable to be replaced by the entering variable (decided by the key row) would be the one with the smallest positive value of the ratio column.
15. **Key Element:** The element at the point of intersection of the key column and the key row is called the key element.
16. **Infeasible Solution:** The solution violating at least one constraint.
17. **Multiple Optimal Solution:** When alternative optimal solution exists.
18. **Optimal Solution:** The best of all feasible solutions.
19. **Unbounded Problem:** A case, where feasible solution extends indefinitely.

20. **Linear Function:** A mathematical expression in which a linear relationship exists amongst various variables.

6.3 Standard form of LP Problem for Simplex Method

In order to develop a general procedure for solving any linear programming (LP) problem, we first introduce the standard form. Let us assume the decision variables as $x_1, x_2, x_3, \ldots. x_n$ such that the objective function (Linear) of these variables assumes an optimum value, when operated under the given constraint of resources. Thus, the standard form of LPP can be written as follows.

Objective Function

Optimise (Maximise or minimise) $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots\ldots\ldots\ldots. c_n x_n$. where c_j (j = 1,2,..................n) are called cost coefficients.

Constraints (Linear)

Subject to,

$a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots a_{1n} x_n = b_1$

$a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots a_{2n} x_n = b_2$

....

....

....

$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots\ldots\ldots\ldots a_{mm} x_n = b_m$.

Where b_i (i = 1, 2..................m) are resources constraints and constants aij (i = 1,2,........................m; j = 1,2,..................n) are called the input output coefficients

Also,

$x_1, x_2 \ldots\ldots\ldots\ldots\ldots\ldots x_n \geq 0$ (non-negative constraint)

The decision variables are required to be non-negative so that they can contribute towards the optimum objective function, which is either maximisation or minimisation type.

6.4 Slack and Artificial Variables

Normally constraints are is the form of inequalities or equalities. When constraints are in the inequality form, we use imaginary variables to remove these inequalities and convert the constraint to equation form to bring in deterministic nature of resources.

When the constraints are of the type $\leq b_i$, then to convert it into equality, we need adding some variable (not constant). This is normally done by adding variables such as S_1, S_2......S_n, which are called slack variables. In physical sense, these slack variables represent unused resources. The slack variables contribute nothing towards the objective function and hence their coefficients in the objective function are to be zeros.

Thus, to illustrate the above concept,

Constraints $a_{i1}x_1 + a_{i2}\, x_2 +..............a_{in}x_n \leq b_i$; $i = 1,2, ...m$ (Canonical form)

Can be written as $a_{i1}\, x_1..................a_{in}\, x_n + S_i = b_i$; $i = 1,2, ...m$

(Standard form)

And the objective function can be written as

Max. or Min. $Z = c_1\, x_1 + c_2\, x_2 + ...\, c_n\, x_n + \text{o}.S_1 + \text{o}.S_2$ +---

Similarly for the constraints of the type $\geq$, the addition of slack variables has to be in the form of subtraction. Thus, equation of constraints can be written as

$a_{i1}\, x_1 + a_{i2}\, x_2 +..............a_{in}\, x_n - S_i = b_i$; $i = 1,2,...................m$

To bring it to the standard form, we add another variable called artificial variable (A_i), as follows:

$a_{i1}\, x_1 + a_{i2}\, x_2 +................a_{in}\, x_n - S_i + A_i = b_i$; $i = 1, 2, 3, ...m.$

This is done to achieve unit matrix for the constraints. But artificial variables can not figure in the solution as there are artificially added variables and have no significance for the objective function. These variables, therefore, are to be removed from the solution.

6.5 Physical Significance of Artificial Variables

As is evident from para 18.4, the standard form of the simplex method can be obtained in the normal course by adding slack variables to the constraints in order to convert inequalities into equalities in case of the maximisation problem, where constraints are of the type ≤. When the constraints are of the type ≥ or =, then equality conversion would need slack variables subtracted. But as per the standard form, this is no more an initial basic feasible solution because simplex method does not accept negative slack variable to start. Hence, just to convert it into initial basic feasible solution, we add Artificial variables (dummy variables) to convert the constraints into standard form with these variables as the basic variables.

By using these additional variables, we apply the simplex method not to the objective function of the original problem, but to a new function containing Artificial variables. This goes contrary to the problem definition itself and hence it becomes imperative to remove these variables from the solution and not to permit their re-entry. It is for this reason that a very large (arbitrary) penalty say M is assigned to each of the artificial variables as its coefficient value in the Objective function. This also indicates the reason for the elimination of artificial variables from the solution, as these variables cannot play any useful part in the objective function.

6.6 Degeneracy

Degeneracy in an LP problem occurs, when one or more of the basic variables assume the value of zero. Normally the number of variables in the solution is the same as the number of constraints. But when the number of positive variables in the solution is less than the number of constraints, the degeneracy occurs. For example, if there are n-variables in an LP problem with m-constraints, there are m basic variables and $(n\text{-}m)$ non-basic variables. All basic variables are required to have positive values. If one or more of the basic variables assume zero value, it will be difficult to find solution to the problem and the solution will become degenerate.

6.7 Modified Simplex Algorithm for Minimisation Problem

We have already discussed the problem of maximisation of an objective function, when constraints were of the type $\leq$. There may be cases when constraints are of $\geq$ or = type. When the objective function is of minimisation type, then simplex method needs some modification.

We have already discussed such a situation earlier, wherein use of slack and artificial variables has been explained.

Some of the important aspects of this case are enumerated below

1. Artificial variables have no economic significance and they are introduced only to bring in the standard form of simplex method. These variables, therefore, need be removed from the solution as soon as they become non-basic.
2. Since these variables are added for computation purpose only, we have to ensure their zero value in the optiomal solution. This can be done by assigning very large penalty ($+M$) for a minimisation problem, so that these do not enter the solution.
3. If artificial variables cannot be removed from the solution, then the solution so obtained is said to be Non-Feasible. This would indicate that the resources of the system are not sufficient to meet the expected demand.
4. Equality Constraints also can be handled by using artificial variables to obtain initial solution.

While discussing the maximisation case, we could obtain the standard form of the simplex algorithm by adding slack variables. But difficulties would arise, when the initial basic feasible solution is not easy to obtain. This can arise under two conditions

(*i*) when the constraints are of the type $\leq$

i.e., $\sum_{j=1}^{n} aij\ xj \leq bi$ where $xj \geq 0$

and right hand side constraints are negative *i.e., bi* ≤ 0. In such cases, even after adding non-negative slack variables Si, the initial solution so obtained will be $Si = - bi$ for some i. Since this violates the non-negative conditions of the slack variables, it will not be a feasible solution.

(*ii*) When constraints are of the type $\geq$

i.e., $\sum_{j=1}^{n} aij\ xj \geq bi$ when $xj \geq 0$

In such cases, we have to add negative slack variables to convert inequalities into equalities *i.e.,* $\sum aij\ xj - Si = bi,$ and we will obtain the initial solution as $-Si = bi$ or $Si = -bi$, which again violates the non-negative constraint of slack variables. In such a situation, we add Artificial Variable Ai to get the initial basic feasible solution. These variables, being of no economic consequences or significance, have to be eliminated from the solution for obtaining optimal solution. We use following methods for eliminating these artificial variables from the solution.

1. Big M-method
2. Dual method
3. Two-phase method

6.7.1 Big M-Method

In this method, we assign the coefficients of the artificial variables, as a very large positive penalty *i.e.,* +M for the minimisation problem. This is, therefore called Big M-method.

The Big M-method for solving LP problem can be adopted as follows:

***Step 1*:** The standard simplex table can be obtained by adding slack and artificial variables. Slack variables are assigned zero coefficients and artificial variables assigned +M coefficients in the objective function.

***Step 2*:** We obtain initial basic feasible solution by assigning zero value to the decision and slack variables.

***Step 3*:** Initial basic feasible solution is obtained in the form of the simplex table as above and then values of $Dj = Cj - Zj$ are calculated. If $Dj \geq 0$, then the optimal solution has been obtained. If $Dj \leq 0$, then we select the largest negative value of Dj and this column becomes the key column indicating the entering variable.

Step 4: Determine the key row as in case of maximisation problem *i.e.,* selecting the lowest positive value of the ratio bi/aij, obtained by dividing the value of quantity bi by corresponding elements of the key column. The key element is thus the point of intersection of the key column and key row. The variable of the key row becomes the outgoing variable.

Step 5: Repeat steps 3 and 4 to ensure optimal solution with no artificial variable in the solution.

If at least one artificial variable is present in the basis with zero value and coefficient of M in each $Cj - Zj$ values is negative, the LP problem has no solution. This basic solution will be treated as degenerate.

It atleast one artificial variable is present in the basis with positive value, and coefficient of M in each $Cj - Zj$ values is non-negative, then LP problem has no optimal basic feasible solution. It is called pseudo-optimum solution. The Big M-method has been demonstrated by solving problem 3.5.

6.7.2 Dual Method

Corresponding to every given linear programming problem, there is another associated LPP, called dual. Primal for given problem and dual are replicas of each other. The difference is while primal is a maximisation problem, the dual would be a minimisation problem or *vice versa*. Thus, Duality plays an important role in business transaction analysis for the following reasons:

1. It has an important economic interpretation.
2. It can help in computer capacity limitations.
3. It can ease the calculations for the minimisation or maximisation problems, whose number of variables is large.

The above reasons come about handy because of following properties of the dual method.

1. Maximisation case can be turned into minimisation and *vice versa* for ease of calculations, as for every primal, there exists a dual.
2. The optimal solution for the dual exists only where there exists an optimal solution to its primal.
3. The dual of the dual is the primal.
4. The dual variables may assume negative values.
5. The value of the objective function of the optimal solution in both the problems is the same.

6.7.3 Economic Interpretations of Dual Variables

The dual variables are termed as shadow prices of the constraints. These can be named as marginal values or the opportunity costs of primal resources. Hence, there is one dual variable for every constraint in the primal and the change in the right hand side is proportional change in the objective function. Because of the same reason of one dual variable for every constraint, the dual being non-zero, indicates full utilisation of that resource. The use of dual concept can be done in making decisions regarding change in resources such as addition, deletion or trade-off of resources. Hence a minimisation problem can also be solved by solving its dual as a maximisation problem.

6.7.4 Two-phase Method

As an alternative to the Big M method, there is yet another method for dealing with LP. involving artificial variables. This is called the Two-phase method and as its name implies, it separates the solution procedure into two phases. In phase I, all the artificial variables are eliminated from the basis. If a feasible solution is obtained in this phase, which has no artificial variable in the basis of the final table, then we proceed to phase II. In this phase, we use the solution from phase I as the initial basic feasible solution and use the simplex method to determine the optimal solution. Illustration of Two-phase method is given by taking problem 3.5.

6.8 Solved Examples

Problem 1:

$$\begin{array}{lrcl} & \text{Maximise } Z & = & 30x_1 + 40x_2 \\ \text{Subject to,} & 60x_1 + 120x_2 & \leq & 12{,}000 \\ & 8x_1 + 5x_2 & \leq & 600 \\ & 3x_1 + 4x_2 & \leq & 500 \\ & x_1, x_2 & \geq & 0 \end{array}$$

Solution:

We convert the inequality into equations by adding slack variables. Above statements can thus be written as follows.

$$60x_1 + 120x_2 + S_1 = 12{,}000$$

$$8x_1 + 5x_2 + S_2 = 600$$

$$3x_1 + 4x_2 + S_3 = 500$$

and $x_1, x_2, S_1, S_2, S_3 \geq 0.$

where S_1, S_2, S_3 are slack variables and objective function is re-written as

$$\text{Max. } Z = 30x_1 + 40x_2 + 0.S_1 + 0.S_2 + 0.S_3.$$

Now there are *five* variables and *three* equations and hence to obtain the solution, any two variables will have to be assigned zero value. Moreover, to get a feasible solution, all the constraints must be satisfied.

To start with, let us assign $x_1 = 0$; $x_2 = 0$ (Both decision variables are assigned zero values)

Hence, $S_1 = 12{,}000$

$S_2 = 600$

$S_3 = 500$

and $Z = 0$

This can be written as initial simplex table I

Unit	*Cj→*	*30*	*40*	*0*	*0*	*0*	*Quantity*	*Ratio*	
Cost Zj	*Variable Mix*	x_1	x_2	S_1	S_2	S_3	*bi*	*bi/aij*	
0	S_1	60	120	1	0	0	12,000	100	→
0	S_2	8	5	0	1	0	600	120	
0	S_3	3	4	0	0	1	500	125	
	Cj - Zj	+30	+40	0	0	0			

↑

Key column

In preparing the table, each column corresponds to one variable and coefficients of the variables in the equations are written above. The initial solution corresponding to unutilised resources S_1, S_2, S_3 is shown at the right under the column "quantity". Since profit associated with each of the slack variables is zero, the total profit at this stage is zero.

Iteration 1

Since *Cj - Zj* is maximum at + 40, *i.e.* profit is more for each unit for x_2 variable, we introduce x_2 into the solution. It is the marked as key column and x_2 becomes the entering variable. Dividing Quantities (bi's) by the corresponding key elements of each row, we obtain the ratio (*bi*/*aij*) column such as for row S_1, it is 12,000 ÷ 120 = 100.

For each unit of x_2, 120 units of S_1, 5 units of S_2 and 4 units of S_3 will have to be utilised. Minimum unutilised capacity of material is 100, it becomes outgoing variable. (This is required to be the least positive value of ratio, which is obtained by dividing the values in *bi* column by the corresponding elements in the key column). Here Key element corresponds to x_2 and S_1 and its value is 120. The least ratio being 100 corresponding to S_1, it becomes the outgoing variable, replaced by x_2.

Now each of the elements of the Key row is divided by Key element to get x_2 row in the new table. Thus we get the key row as follows:

Unit cost	*Variable Mix*	x_1	x_2	S_1	S_2	S_3	*bi*
40	x_2	$60/120$	$120/120$	$1/120$	0	0	$12{,}000/120$
i.e. 40	x_2	$1/2$	1	$1/120$	0	0	100

In order to obtain the corresponding values of the table, we follow the relationship as follows:

New row = old row - Reduced key row element × corresponding key column element

for S_2 row,	S_2	$8 - 5 \times 1/2$	$5-5\times1$	$0 - 5 \times 1/120$	$1-5\times0$	$0-5\times0$	$600-5\times100$
or,	S_2	$= 1\frac{1}{2}$	$= 0$	$= -5/20$	$= 1$	$= 0$	$= 100$
Similarly for S_3,	S_3	$3 - 4 \times 1/2$	$4-4\times1$	$0 - 4 \times 1/120$	$0-4\times0$	$1-4\times0$	$500-4\times100$
or,	S_3	$= 1$	$= 0$	$= -1/30$	$= 0$	$= 1$	$= 100$

Arranging these values into the simplex table, we obtain:

Revised Simplex Table II

Unit Cost	Dj →	30	40	0	0	0	Quantity	Ratio	
Ƶj ↓	*Variable mix* ↓	x_1	x_2	S_1	S_2	S_3	*bi*	*bi/aij*	
40	x_2	$\frac{1}{2}$	1	$\frac{1}{120}$	0	0	100	200	
0	s_2	$\frac{11}{2}$	0	$-\frac{5}{20}$	1	0	100	$\frac{200}{11}$	→
0	s_3	1	0	$-\frac{1}{30}$	0	1	100	100	
	Δ*j* = *Dj* - *Ƶj*	+10	0	$-\frac{1}{3}$	0	0			

↑

This solution suggests that by producing 100 units of x_2 variable, we get an idle capacity of 100 units pertaining to S_2 and S_3 each. Since, Maximum positive value for *Dj* - *Ƶj* being 10 corresponds to x_1, it becomes the entering variable. Ratios are calculated by dividing *bi* by *aij i.e.,*. corresponding key elements and are given in the last column. The minimum positive value of ratio being 200/11, pertaining to S_2, we replace S_2 by x_1.

Iteration 2

Key column being that for x_1 and key row for S_2, we get the key element as 11/2 and by dividing S_2 row by the Key element, we get x_1 row in the new table as follows:

Revised Simplex Table III

Ƶj ↓	*Cj*→	*30*	*40*	*0*	*0*	*0*	*bi*
	Variable mix	x_1	x_2	S_1	S_2	S_3	
40	x_2	0	1	$\frac{16}{1,320}$	$-\frac{1}{11}$	0	$\frac{1,000}{11}$
30	x_1	1	0	$-\frac{1}{132}$	$\frac{2}{11}$	0	$\frac{200}{11}$
0	S_3	0	0	$-\frac{34}{1,320}$	$-\frac{2}{11}$	1	$\frac{900}{11}$
Cj - *Ƶj*		0	0	$-\frac{34}{132}$	$-\frac{20}{11}$	0	$\frac{46,000}{11}$

Other rows are correspondingly obtained as in table I.

Since all the values of *Cj* - *Ƶj* are either negative or zeros, it is an optimal solution.

Hence, the optimal mix would be

Quantity of item 1 $(x_1) = {}^{200}\!/_{11}$

Quantity of item 2 $(x_2) = {}^{1,000}\!/_{11}$

Total profit $= {}^{46,000}\!/_{11}$

By this iterative process, we obtain optimal solution of a maximising problem and iteration stops when values of $Cj - Zj \leq 0$. This optimality test is carried out at every stage to reach optimal solution.

Since slack variable S_3 remains in the solution, this indicates unutilised resource of third constraint.

Problem 2:

Use the simplex method to solve the following LP problem.

$$\text{Maximise } Z = 3x_1 + 5x_2 + 4x_3$$

subject to the constraints,

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Step 1. The problem is available in the mathematical form.

Step 2. Set up initial solution.

Introducing slack variables for conversion of inequality constraints to equality, the modified standard simplex form becomes

Objective function:

$$\text{Max. } Z = 3x_1 + 5x_2 + 4x_3 + 0.S_1 + 0.S_2 + 0.S_3$$

and constraints, $2x_1 + 3x_2 + S_1 = 8$

$$2x_2 + 5x_3 + S_2 = 10$$

$$3x_1 + 2x_2 + 4x_3 + S_3 = 15$$

and $$x_1, x_2, x_3, S_1, S_2, S_3 \geq 0$$

Since all *bi*'s are positive, we choose initial solution as,

$x_1 = x_2 = x_3 = 0$ and $S_1 = 8$, $S_2 = 10$, $S_3 = 15$ and $Ƶ = 0$

Initial Simplex Table I

Ƶj ↓	*Dj*→	3	5	4	0	0	0			
	Variable mix	x_1	x_2	x_3	S_1	S_2	S_3	*bi*	*bi/aij*	
0	S_1	2	3	0	1	0	0	8	$\frac{8}{3}$	→
0	S_2	0	2	5	0	1	0	10	5	
0	S_3	3	2	4	0	0	1	15	$\frac{15}{2}$	
$\Delta j = Dj - Ƶj$		3	5	0	0	0	0			
			↑							

Step 3. Optimality Test—Since the elements in the Index row (D*j*) are not negative or zero, the solution is not optimal.

Step 4. For improvement of the solution, we proceed as follows:

(*a*) Identify Entering Variable—Maximum positive value of index row is 5, hence key column is as marked by arrow → corresponding to x_2 and hence x_2 becomes the entering variable.

(*b*) Identify Outgoing Variable—Divide column bi by the corresponding elements of key column, i.e. $\frac{8}{3}$, 5 and $\frac{15}{2}$. The lowest positive value of the ratios is 8/3, corresponding to row for S_1 and hence S_1 becomes the outgoing variable.

Step 5. Identify Key Element—Point of intersection of x_2 and S_1 indicates the value as 3, which is the key element.

Step 6. Iteration 1 for Improved solution

(*a*) Replace S_1 by x_2 with corresponding value *Ƶj* modified as 5.

(*b*) Key row is modified by dividing the old row by key element 3.

(*c*) Modify other rows of S_2 and S_3 as per the relationship indicating under Algorithm procedure.

(*d*) Values of bi's are also modified as per the same procedure

(*e*) Evaluate $\Delta j = Cj - Ƶj$ for testing optimality.

Revised Simplex Table II

Ƶj↓	Cj→	3	5	4	0	0	0			
	Variable mix	x_1	x_2	x_3	S_1	S_2	S_3	*bi*	*bi/aij*	
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	∞	
0	S_2	$-\frac{4}{3}$	0	5	$\frac{-2}{3}$	1	0	$\frac{14}{3}$	$\frac{14}{15}$	→
0	S_3	$\frac{5}{3}$	0	4	$\frac{-2}{3}$	0	1	$\frac{29}{3}$	$\frac{29}{12}$	
$\Delta j = Cj - Ƶj$		$-\frac{1}{3}$	0	4	$\frac{-5}{3}$	0	0			
					↑					

Step 7. Since all values of Index row D*j* are not negative or zeros, the solution is not optimal. Hence proceed to Step 8.

Step 8. Repeating Step 4 to 7, we get revised simplex table on iteration 2 for x_3 as entering variable and S_2 as outgoing variable.

Simplex Table III

Ƶj↓	Cj→	3	5	4	0	0	0			
	Variables mix	x_1	x_2	x_3	S_1	S_2	S_3	*bi*	*bi/aij*	
5	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	4	
4	x_3	$\frac{-4}{15}$	0	1	$\frac{-2}{15}$	$\frac{1}{5}$	0	$\frac{14}{15}$	$\frac{-7}{2}$	
0	S_3	$\frac{41}{15}$	0	0	$\frac{2}{15}$	$\frac{-4}{5}$	1	$\frac{89}{15}$	$\frac{89}{41}$	®
$\Delta j = Cj - Ƶj$		$\frac{11}{15}$	0	0	$\frac{-17}{15}$	$\frac{-4}{5}$	0			
		↑								

Step 9. (Revised) Since all values of Index row are not negative or zeros, the solution is not optimal. Hence, we repeat Step 4 to 7.

(*a*) Entering variable is x_1 (highest positive value column).

(*b*) S_3 is outgoing variable due to lowest positive value row ratio.

(*c*) Revised simplex table is as given below.

Revised Simplex Table IV

Ƶj¯	Cj®	3	5	4	0	0	0	
	Variable mix¯	x_1	x_2	x_3	S_1	S_2	S_3	bi
5	x_2	0	1	0	$^{15}/_{41}$	$^{8}/_{41}$	$^{-10}/_{41}$	$^{50}/_{41}$
4	x_3	0	0	1	$^{-6}/_{41}$	$^{5}/_{41}$	$^{4}/_{41}$	$^{62}/_{41}$
3	x_1	1	0	0	$^{-2}/_{41}$	$^{-12}/_{41}$	$^{15}/_{41}$	$^{89}/_{41}$
	Δj = Cj - Ƶj		0	0	0	$-^{45}/_{41}$	$^{-24}/_{41}$	$^{-11}/_{41}$

Since all the values of Δ*j* row are either negative or zeros, the solution is optimal.

Hence, optimal solution is $x_1 = {}^{89}/_{41}$

$x_2 = {}^{50}/_{41}$

$x_3 = {}^{62}/_{41}$

and max. $Ƶ = {}^{765}/_{41}$

Problem 3:

A marketing manager wishes to allocate his annual advertising budget of ₹ 20,000 to two media vehicles A and B. The unit cost of message in media A is ₹ 1,000 and that in B it is ₹ 1500. Media A is a monthly magazine and not more than one insertion is desired in one issue. Atleast 5 messages should appear in media B. The expected effected audience for unit messages in the media A is 40,000 and for media B is 55,000.

(*i*) Develop a mathematical model.

(*ii*) Solve it for maximising the total effective audience.

Solution:

Step 1. The mathematical model is as follows:

$$\text{Max. } Z = 40{,}000x_1 + 55{,}000x_2$$

subject to, $1{,}000x_1 + 1{,}500x_2 \leq 20{,}000$

$x_1 \leq 12$

$x_2 \geq 5$ or $-x_2 \leq -5$

and $x_1, x_2 \geq 0$

where x_1 = annual number of insertions for medium A and

x_2 = annual number of insertions for medium B.

The standard simplex form can be written as

$$\text{Max. } Z = 40{,}000x_1 + 55{,}000x_2 + 0.S_1 + 0.S_2 + 0.S_3$$

subject to, $1{,}000x_1 + 1{,}500x_2 + S_1 = 20{,}000$

$x_1 + S_2 = 12$

$-x_2 + S_3 = -5$

and $x_1, x_2, S_1, S_2, S_3 \geq 0$

where S_1, S_2 and S_3 are slack variables introduced to convert inequalities into equalities.

Step 2. For initial solution, we use $x_1 = 0$, $x_2 = 0$ and prepare the initial simplex table.

Simplex Table I

Zj↓	*Cj* →	*40,000*	*55,000*	*0*	*0*	*0*		*Ratio*	
	Variable mix↓	x_1	x_2	S_1	S_2	S_3	*bi*	*bi/aij*	
0	S_1	1,000	1,500	1	0	0	20,000	$^{40}/_{3}$	
0	S_2	1	0	0	1	0	12	∞	
0	S_3	0	-1	0	0	1	-5	5	→
$\Delta j = Cj - Zj$	40,000	55,000	0	0	0				
			↑						

Step 3. Since values of Δj are zeros or positive, the solution is not optimal.

Step 4. The initial solution indicate highest Δj = 55,000 and hence x_2 becomes the entering variable and this column as key column. Lowest ratios variable S_3 becomes the outgoing variable. Thus S_3 is to be replaced by x_2 in the next improved iteration.

Step 5. As per the above step, the key element turns out to be -1.

Step 6. New simplex table thus can be written as

Simplex Table II

Zj↓	*Cj*→ *Variable mix*↓	*40,000* x_1	*55,000* x_2	*0* S_1	*0* S_2	*0* S_3	*bi*	*Ratio* *bi/aij*	
0	S_1	1,000	0	1	0	1,500	12,500	$^{25}\!/_3$	→
0	S_2	1	0	0	1	0	12	∞	
55,000	x_2	0	1	0	0	-1	5	-5	
Δj = C*j* - Z*j*		40,000	0	0	0	55,000			
						↑			

Step 7. Testing for optimality, we find that solution is not optimal yet, because values of D*j* are either positive or zeros. Proceed to step 8.

Step 8. Having obtained the condition that outgoing variable S_1 is to be replaced by the entering variable S_3, we get the revised simplex table as follows : (Key element being 1,500).

Simplex Table III

Zj↓	*Cj* → *Variable mix*↓	*40,000* x_1	*55,000* x_2	*0* S_1	*0* S_2	*0* S_3	*bi*	*bi/aij*	
0	S_3	$^2\!/_3$	0	$^1\!/_{1,500}$	0	1	$^{25}\!/_3$	$^{25}\!/_2$	
0	S_2	1	0	0	1	0	12	12	→
55,000	x_2	$^2\!/_3$	1	$^1\!/_{1,500}$	0	0	$^{40}\!/_3$	20	
Δj = *Cj* - *Zj*		$^{10,000}\!/_3$	0	$^{-110}\!/_3$	0	0			
		↑							

The solution is still not optimal due to positive value of Δj. Hence maximum Δj indicates x_1 column as key column and x_1 becomes the entering variable. Calculating ratios, we find the minimum positive value for S_2 and hence S_2 is the key row. The variable S_2 thus becomes the outgoing variable with 1 as the key element.

Simplex Table IV

Z_j	$C_j \rightarrow$	40,000	55,000	0	0	0	
	Variable mix↓	x_1	x_2	S_1	S_2	S_3	bi
0	S_3	0	0	$1/1{,}500$	$-2/3$	1	$1/3$
40,000	x_1	1	0	0	1	0	12
55,000	x_2	0	1	$1/1{,}500$	$-2/3$	0	$16/3$
$\Delta j = C_j - Z_j$		0	0	$-110/3$	$-10{,}000/3$	0	

Since all the values of D*j* are either zeros or negative, we have obtained the optimal solution.

Thus $x_1 = 12, x_2 = 16/3$ and

$Z = 20{,}000/3$ is the optimal solution

$S_3 = 1/3$ indicates unutilised capacity for medium B.

Problem 4:

$$\text{Minimise Cost } Z = 60x_1 + 80x_2$$

Subject to, $20x_1 + 30x_2 \geq 900$ (Constraint I)

$40x_1 + 30x_2 \geq 1200$ (Constraint II)

and $x_1, x_2 \geq 0$ (Non-negative Constraint)

Solution:

To convert the problem into standard simplex form, the inequalities are to be converted into equalities by inserting slack variable S_1 and S_2 and artificial variables A_1 and A_2.

$$20x_1 + 30x_2 - S_1 + A_1 = 900$$

$$40x_1 + 30x_2 - S_2 + A_2 = 1{,}200$$

and objective function can be written as

$$\text{Min. } Z = 60x_1 + 80x_2 + 0.S_1 + 0.S_2 + M.A_1 + M.A_2$$

Initial Simplex table (non-optimal) can be written by setting

$$x_1 = x_2 = S_1 = S_2 = 0$$

Simplex Table I

Zj	Cj→ Variables↓	60 x_1	80 x_2	0 S_1	0 S_2	M A_1	M A_2	Quantity bi	Ratio bi/aij	
M	A_1	20	30	-1	0	1	0	900	45	
M	A_2	40	30	0	-1	0	1	1,200	30	→
	$\Delta j = Cj - Zj$	60-60M	80-60M	M	M	0	0			
	↑									

Since D*j* = 60-60M is the lowest, x_1 become the entering variable. Similarly Ratio *bi*/*aij* = 30 is lowest positive value. Hence A_2 qualifies as outgoing or leaving variable.

Simplex Table II

Zj	Cj→ Variables↓	60 x_1	80 x_2	0 S_1	0 S_2	M A_1	M A_2	Quantity bi	Ratio bi/aij	
M	A_1	0	15	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	300	20	→
60	x_1	1	$\frac{3}{4}$	0	$-\frac{1}{40}$	0	$\frac{1}{40}$	30	40	
	$\Delta j = Cj - Zj$	0	35-15M	M	$\frac{(3-M)}{2}$	0	$\frac{(M-3)}{2}$			
			↑							

From the above table, it can be seen that x_2 becomes the entering variable and A_1 the leaving variable. The modified simplex table can be written as follows:

Simplex Table III

Zj	Dj→ Variables↓	60 x_1	80 x_2	0 S_1	0 S_2	M A_1	M A_2	bi
80	x_2	0	1	$-\frac{1}{15}$	$\frac{1}{30}$	$\frac{1}{15}$	$-\frac{1}{30}$	20
60	x_1	1	0	$\frac{1}{20}$	$-\frac{1}{20}$	$-\frac{1}{20}$	$\frac{1}{20}$	15
	$\Delta j = Dj - Zj$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$M-\frac{7}{3}$	$M-\frac{1}{3}$	

Since all the values of D*j* are non-negative, we have reached the optimal solution of the problem.

Hence the optimal solution is given by

$x_1 = 15$ and $x_2 = 20$

and Minimum $Z = 2{,}500$.

Problem 5:

Write a dual for a given primal.

Primal is Max. $Z = 30x_1 + 20x_2$

Subject to, $2x_1 + 3x_2 \leq 45$

$4x_1 + 5x_2 \leq 85$

$x_1, x_2 \geq 0$

Solution:

Taking y_1, y_2 as new variables for two constraints.

In Dual problem, it would be

Min. $Z = 45y_1 + 85y_2$

Subject to, $2y_1 + 4y_2 \geq 30$

$3y_1 + 5y_2 \geq 20$

and $y_1, y_2 \geq 0$

In case, inequalities are not in the right direction, they should be converted so. Like in a minimisation problem, if the inequality is $2x_1 - x_2 \leq 6$, it should be changed to $- 2x_1 + x_2 \geq -6$ and problem is solved in the normal manner. In Dual, the quantities become the objective function and objective function gets into the constraints.

Problem 6:

Apply Duality concept for the following minimisation problem.

Minimise $G = 40y_1 + 24y_2$

Subject to, $20y_1 + 50y_2 \geq 4{,}800$

$80y_1 + 50y_2 \geq 7{,}200$

$y_1, y_2 \geq 0$

Its dual can be written as follows:

$$\text{Max. Ƶ} = 4{,}800x_1 + 7{,}200x_2$$

Subject to, $20x_1 + 80x_2 \leq 40$

$50x_1 + 50x_2 \leq 24$

$x_1, x_2 \geq 0$

This can now be solved as normal maximisation problem by standard simplex method.

Solution:

We solve the dual by simplex method by adding slack variables to convert the inequalities into equalities. Thus

$$\text{Max. Ƶ} = 4{,}800x_1 + 7{,}200x_2 + 0.S_1 + 0.S_2$$

Subject to, $20x_1 + 80x_2 + S_1 = 40$

$50x_1 + 50x_2 + S_2 = 24$

and $x_1, x_{2,} S_1, S_2 \geq 0$

The initial non-optimal solution is obtained by setting up

$$x_1 = x_2 = \text{Ƶ} = 0$$

Initial Simplex table can be written as follows:

Simplex Table I

Ƶj ↓	*Cj*→	*4,800*	*7,200*	*0*	*0*	*bi*	*bi/aij*	
	Variables↓	x_1	x_2	S_1	S_2			
0	S_1	20	80	1	0	40	½	
0	S_2	50	50	0	1	24	12⁄25	→
D*j* = *Cj* - *Ƶj*		4,800	7,200	0	0			
		↑						

From the table, S_2 become the leaving variable and x_2 as entering variable. Table gets converted as

Simplex Table II

Zj‾	Cj →	4,800	7,200	0	0	bi
	Variables↓	x_1	x_2	S_1	S_2	
0	S_1	-60	0	1	$-8/5$	$8/5$
7,200	x_2	1	1	0	$1/50$	$12/25$
$\Delta j = Cj - Zj$		-2,400	0	0	-144	

This has thus become the optimal solution, as D*j* has no positive values.

Hence, $x_1 = 0, x_2 = 12/25$

and $\text{Max. } Z = 7200 \times 12/25 = 3{,}456$

or, $\text{Min. } G = 3{,}456.$

Problem 7:

$$\text{Min. } Z = 40x_1 + 24x_2$$

$$20x_1 + 50x_2 \geq 4{,}800$$

$$80x_1 + 50x_2 \geq 7{,}200$$

$$x_1, x_2 \geq 0$$

Phase I

Initial solution is for

$\text{Min. } Z = 0.x_1 + 0.x_2 + 0.S_1 + 0.S_2 + A_1 + A_2$

Simplex Table I

Zj	Cj →	0	0	0	0	1	1	bi	bi/aij	
	Variables ↓	x_1	x_2	S_1	S_2	A_1	A_2			
1	A_1	20	50	-1	0	1	0	4,800	240	
1	A_2	80	50	0	-1	0	1	7,200	90	→
	$\Delta j = (Cj - Zj)$	-100	-100	1	1	0	0			
		↑								

x_1 thus becomes the entering variable and A_2 the outgoing variable.

Simplex Table II

Ƶj	Cj →	0	0	0	0	1	1			
	Variables ↓	x_1	x_2	S_1	S_2	A_1	A_2	bi	bi/aij	
1	A_1	0	$\frac{75}{2}$	-1	$\frac{1}{4}$	1	$-\frac{1}{4}$	3,000	80	→
0	x_1	1	$\frac{5}{8}$	0	$-\frac{1}{80}$	0	$\frac{1}{80}$	90	144	
	Δj	0	$-\frac{75}{2}$	1	$-\frac{1}{4}$	0	$\frac{5}{4}$			
			↑							

Now x_2 becomes the entering variable replacing A_1

Simplex Table III

Ƶj	Cj →	0	0	0	0	1	1	
	Variables ↓	x_1	x_2	S_1	S_2	A_1	A_2	bi
0	x_2	0	1	$-\frac{2}{75}$	$\frac{1}{150}$	$\frac{2}{75}$	$-\frac{1}{150}$	80
0	x_1	1	0	$\frac{1}{60}$	$-\frac{1}{60}$	$\frac{1}{60}$	$\frac{1}{60}$	40
	Δj	0	0	0	0	1	1	

There being no negative values of D*j*, this is an optimal solution.

Phase II

The simplex table III is reproduced after replacing *Cj* row by respective coefficients from original problem and deleting columns A_1 and A_2. The values of *Ƶj* are also replaced from the original problem.

Revised Simplex Table III

	Cj→	40	24	0	0			
Ƶj	Variables ↓	x_1	x_2	S_1	S_2	bi	bi/aij	
24	x_2	0	1	$-\frac{2}{75}$	$\frac{1}{150}$	80	-3,000	
40	x_1	1	0	$\frac{1}{60}$	$-\frac{1}{60}$	40	2,400	→
	Δj	0	0	$-\frac{2}{75}$	$\frac{38}{75}$			
				↑				

Thus S_1 enters the solution replacing x_1, the outgoing variable.

Simplex Table V

	Cj →	*40*	*24*	*0*	*0*	
Ƶj	*Variables* ↓	x_1	x_2	S_1	S_2	*bi*
24	x_2	$\frac{8}{5}$	1	0	$-\frac{1}{150}$	144
0	S_1	60	0	1	-1	2,400
	Δ*j*	$\frac{8}{5}$	0	0	$\frac{12}{25}$	

Since there are no negative values for Δ*j*, the solution is optimal.

Thus optimal solution is given by $x_1 = 0, x_2 = 144$.

and Min. Ƶ = 3,456, same as obtained in problem 18.7.

Problem 8:

Obtain the dual of the following LP Problem

Min. $Ƶ = 15x_1 + 20x_2$

Subject to, $3x_1 + 2x_2 \geq 20$

$x_1 + 3x_2 \geq 15$

$+2x_1 - x_2 \leq 6$ or, $-2x_1 + x_2 \geq -6$

and $x_1, x_2 \geq 0$

Solution:

DualMax. $G = 20y_1 + 15y_2 - 6y_3$

Subject to, $3y_1 + y_2 - 2y_3 \leq 15$

$2y_1 + 3y_2 + y_3 \leq 20$

and $y_1, y_2, y_3 \geq 0$

Problem 9:

One unit of product A contributes ₹ 7 and requires 3 units of raw material and 2 hrs. of labour. One unit of product B contributes ₹ 5 and requires two units of raw material and one hour of labour. Availability of raw material at present is 48 units and that of labour as 40 hours.

(*a*) Formulate it as linear programming problem.

(*b*) Write its dual

(*c*) Solve the dual by simplex method and find the optimal product mix and shadow prices of the raw material and labour.

Solution:

(*a*) The LPP corresponding to the given information is

$$\text{Max. } Z = 7x_1 + 5x_2$$

$$\text{Subject to, } \quad 3x_1 + x_2 \leq 48$$

$$2x_1 + x_2 \leq 40$$

$$x_1, x_2 \geq 0$$

(*b*) The dual of this problem is given by

$$\text{Min. } G = 48y_1 + 40y_2$$

$$\text{Subject to, } \quad 3y_1 + 2y_2 \geq 7$$

$$y_1 + y_2 \geq 5$$

$$y_1, y_2 \geq 0$$

(*c*) Introducing slack and artificial variables, we get

$$\text{Min. } G = 48y_1 + 40y_2 + 0.S_1 + 0.S_2 + MA_1 + MA_2$$

$$\text{Subject to, } \quad 3y_1 + 2y_2 - S_1 + A_1 = 7$$

$$y_1 + y_2 - S_2 + A_2 = 5$$

$$y_1, y_2, S_1, S_2, A_1, A_2 \geq 0$$

Solution by Simplex Method (Big M Method)

Simplex Table 1

Ƶj	Cj→	48	40	0	0	M	M	Qty.	Ratio	
↓	Variables¯	y_1	y_2	S_1	S_2	A_1	A_2	bi	bi/aij	
M	A_1	3	2	-1	0	1	0	7	7/3	→
M	A_2	1	1	0	-1	0	1	5	5	
	Δj	(48-4M)	(40-3M)	M	M	0	0			
		↑								

Simplex Table II

	Cj→	48	40	0	0	M	M			
Ƶj	Variables	y_1	y_2	S_1	S_2	A_1	A_2	bi	bi/aij	
48	y_1	1	$\frac{2}{3}$	$\frac{-1}{3}$	0	$\frac{1}{3}$	0	$\frac{7}{3}$	$\frac{7}{2}$	→
M	A_2	0	$\frac{1}{3}$	$\frac{1}{3}$	-1	$\frac{-1}{3}$	1	$\frac{8}{3}$	8	
	Δj	0	$\left(8-\frac{M}{3}\right)$	16	M	$\left(\frac{4M}{3}-16\right)$	0			
			↑							

Simplex Table III

	Cj→	48	40	0	0	M	M			
Ƶj	Variables	y_1	y_2	S_1	S_2	A_1	A_2	bi	bi/aij	
40	y_2	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	$\frac{1}{2}$	0	$\frac{7}{2}$	-7	
M	A_2	$\frac{-1}{2}$	0	$\frac{1}{2}$	-1	$\frac{-1}{2}$	1	$\frac{3}{2}$	3	→
	Δj	$\left(\frac{M}{2}-12\right)$	0	$\left(20-\frac{M}{2}\right)$	M	$\left(\frac{3M}{2}-20\right)$	0			
				↑						

Simplex Table IV

	Cj→	48	40	0	0	M	M	
Ƶj	Variables	y_1	y_2	S_1	S_2	A_1	A_2	bi
40	y_2	1	1	0	-1	0	1	5
0	S_1	-1	0	1	-2	-1	2	3
	Δj	8	0	0	40	M	M–40	

All values of Δ*j* being positive or zero, the solution is optimal.

Hence $y_1 = 0$, $y_2 = 5$ *i.e.,* Shadow price of raw material is NIL and that of the labour is ₹ 5 per hour.

Problem 10:

The simplex table for a maximisaion problem of linear programming is given here.

Zj	*xj*	x_1	x_2	S_1	S_2	*Quantity (bi)*
5	x_2	1	1	1	0	10
0	S_2	1	0	-1	1	3
	Cj	4	5	0	0	
	Zj	5	5	5	0	50
	$\Delta j = Cj - Zj$	-1	0	-5	0	

Answer the following questions, giving reasons in brief;

(*a*) Is this solution optimal?

(*b*) Are there more than one optimal solutions?

(*c*) Is this solution degenerate?

(*d*) Is this solution feasible?

(*e*) If S_1 is slack in machine A (in hours/week) and S_2 is slack in machine B (in hours/week), which of these machines is being used to the full capacity, when producing according to this solution.

(*f*) A customer would like to have one unit of product x_1 and is willing to pay in excess of the normal price in order to get it. How much should the price be increased in order to ensure no reduction in profits?

(*g*) How many units of the two products x_1 and x_2 are being produced according to this solution and what is the total profit?

(*h*) Machine A (associated with slack S_1, in hours per week) has to be shut down for repairs for 2 hours next week. What will be the effect on profits?

(*i*) How much would you be prepared to pay for another hour (per week) of capacity each on machine A and machine B?

Solution:

From the above table, we find

(*a*) Yes, the given solution is optimal because all the Δ*j* are less than, or equal to zero.

(*b*) No, because for each of the non-basic variable x_1 and S_1, the Δ*j* is negative.

(*c*) The solution is not degenerate because none of the basic variable has a zero value.

(*d*) Yes, because the given solution has no artificial variable in the basis.

(*e*) Machine A is being used to the full capacity because, the slack variable S_1 corresponding to it has a zero value in the solution.

(*f*) Δ*j* for x_2 being -1, production of each unit of x_1 would cause a reduction of 1 rupee. Thus the price for x_1 should be increased by at least one rupee to ensure non-reduction of profits.

(*g*) According to the given solution, none of x_1 and 10 units x_2 are being produced. The total profit is 4 × 0 + 5 × 10 = 50.

(*h*) The shadow price of hours on machine *A*, obtained from D*j* is ₹ 5/hr. Thus, reduction in profit of a 2 hour shut down = 2 × 5 = **₹ 10**.

(*i*) The shadow prices of hours on machine A and B being ₹ 5 and ₹ 0, respectively, these are the maximum prices one would be prepared to pay for another hour of capacity for these two machines.

Problem 11:

Find the dual problem for the following:

$$\text{Min. } Z = 5x_1 - 6x_2 + 4x_3$$

Subject to,
$$\begin{aligned} 3x_1 + 4x_2 + 6x_3 &\geq 9 \\ x_1 + 3x_2 + 2x_3 &\geq 5 \\ 7x_1 - 2x_2 - x_3 &\leq 10 \end{aligned}$$

$$x_1 - 2x_2 + 4x_3 \geq 4$$

$$2x_1 + 5x_2 - 3x_3 \geq 3$$

$$x_1, x_2, x_3 \geq 0$$

Solution:

Primal in the normal canonical form can be written as

Min. $Z = 5x_1 - 6x_2 + 4x_3$

Subject to, $3x_1 + 4x_2 + 6x_3 \geq 9$

$x_1 + 3x_2 + 2x_3 \geq 5$

$-7x_1 + 2x_2 + x_3 \geq -10$

$x_1 - 2x_2 + 4x_3 \geq 4$

$2x_1 + 5x_2 - 3x_3 \geq 3$

and $x_1, x_2, x_3 \geq 0$

Dual

Max. $G = 9y_1 + 5y_2 - 10y_3 + 4y_4 + 3y_5$

Subject to, $3y_1 + y_2 - 7y_3 + y_4 + 2y_5 \leq 5$

$4y_1 + 3y_2 + 2y_3 - 2y_4 + 5y_5 \leq -6$

$6y_1 + 2y_2 + y_3 + 4y_4 - 3y_5 \leq 4$

and $y_1, y_2, y_3, y_4, y_5 \geq 0$

Problem 12:

Solve the following problem by simplex method.

Max. $Z = 8x_1 + 16x_2$

Subject to, $x_1 + x_2 \leq 200$

$x_2 \leq 125$

$3x_1 + 6x_2 \leq 900$

and $x_1, x_2 \geq 0$

Solution:

Adding slack variables and arranging in the simplex table, we get

Initial Simplex Table I

Ƶj	Cj→	8	16	0	0	0			
	Variables ↓	x_1	x_2	S_1	S_2	S_3	bi	bi/aij	
0	S_1	1	1	1	0	0	200	200	
0	S_2	0	1	0	1	0	125	125	→
0	S_3	3	6	0	0	1	900	150	
	Δj = Cj - Ƶj	8	16	0	0	0			
		↑							

Simplex Table II

	Cj→	8	16	0	0	0			
Ƶj	Variables ↓	x_1	x_2	S_1	S_2	S_3	bi	bi/aij	
0	S_1	1	0	1	-1	0	75	75	
16	x_2	0	1	0	1	0	125	∞	
0	S_3	3	0	0	-6	1	150	50	→
	Δj = Cj - Ƶj	+8	0	0	-16	0			
		↑							

Simplex Table III

	Cj→	8	16	0	0	0	
Ƶj	Variables ↓	x_1	x_2	S_1	S_2	S_3	bi
0	S_1	0	0	1	1	$-\frac{1}{3}$	25
16	x_2	0	1	0	1	0	125
8	x_1	1	0	0	-2	$\frac{1}{3}$	50
	Δj = Cj - Ƶj	0	0	0	0	$-\frac{8}{3}$	

The values of Δj being zero or negative, suggest that solution is optimal and Ƶ = 2,400 for x_1 = 50 and x_2 = 125. S_1 indicates surplus capacity.

Problem 13:

Food A contains 20 units of vitamin X and 40 units of vitamin Y per gram. Food B contains 30 units each of vitamin X and Y. The daily minimum human requirements of vitamin X and Y are 900 and 1200 units respectively. How many grams of each type of food should be consumed so as to minimise the cost, if food A costs 60 paise per gram and food B costs 80 paise per gram.

Solution:

LPP formulation is as follows

Min. Ƶ = $60x_1 + 80x_2$(Total Cost)

Subject to, $20x_1 + 30x_2 \geq 900$ (Vitamin X Constraint)

$40x_1 + 30x_2 \geq 1{,}200$ (Vitamin Y Constraint)

and $x_1, x_2 \geq 0$

Adding slack and artificial variables, we get

Min. Ƶ $= 60x_1 + 80x_2 + 0.S_1 + 0.S_2 + M.A_1 + M.A_2$

Subject to, $20x_1 + 30x_2 - S_1 + A_1 = 900$

$40x_1 + 30x_2 - S_2 + A_2 = 1{,}200$

and $x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$

Initial non-optimal solution is written as follows:

Initial Simplex Table I

Ƶj	*Cj→*	*60*	*80*	*0*	*0*	*M*	*M*			
	Variables↓	x_1	x_2	S_1	S_2	A_1	A_2	*bi*	*bi/aij*	
M	A_1	20	30	-1	0	1	0	900	45	
M	A_2	40	30	0	-1	0	1	1,200	30	→
	Δ*j*	60-60*M*	80-60*M*	*M*	*M*	0	0			
		↑								

Simplex Table II

	Cj→	*60*	*80*	*0*	*0*	*M*	*M*			
Ƶj	*Variables↓*	x_1	x_2	S_1	S_2	A_1	A_2	*bi*	*bi/aij*	
M	A_1	0	15	-1	$\frac{1}{2}$	1	$-\frac{1}{2}$	300	20	→
60	x_1	1	$\frac{3}{4}$	0	$-\frac{1}{40}$	0	$-\frac{1}{40}$	30	40	
	Δ*j*	0	(35-15*M*)	*M*	$\frac{(3-M)}{2}$	0	$\frac{(M-3)}{2}$			
			↑							

Simplex Table III (Replacing A_1 by x_2)

	Cj→	*60*	*80*	*0*	*0*	*M*	*M*	
Ƶ*j*	*Variables* ↓	x_1	x_2	S_1	S_2	A_1	A_2	*bi*
80	x_2	0	1	$-1/15$	$1/30$	$1/15$	$-1/30$	20
60	x_1	1	0	$1/20$	$-1/20$	$-1/20$	$1/20$	15
	Δ*j*	0	0	$1/3$	$1/3$	$(M-7/3)$	$(M-1/3)$	

Since all values of Δ*j* are positive or zeros, we have reached the optimal solution.

Thus 15 gram of food A and 20 gram of food B should be purchased for a total cost of ₹ **25**.

Problem 14:

Solve the following problem.

Max. $Z = 28x_1 + 30x_2$

Subject to,

$$6x_1 + 3x_2 \leq 18$$

$$3x_1 + x_2 \leq 8$$

$$4x_1 + 5x_2 \leq 30$$

$$x_1; x_2 \geq 0$$

Solution:

Graphically, it can be represented like this (Refer Fig. 6.1).

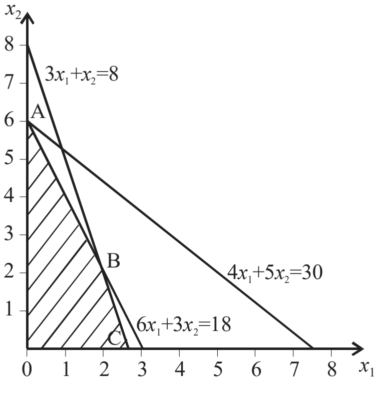

Fig. 6.1

The feasible region is bounded by the polygon OABC. The extreme points are evaluated here

Point	x_1	x_2	Z
O	0	0	0
A	0	6	180 (Max.)
B	2	2	116
C	$8/3$	0	$224/3$

Thus the optimal solution lies at A. A careful look would show that at this point, two constraints are satisfied simultaneously and one of the decision variable x_1 is equal to zero. This means that slack variable in respect of two constraints shall be equal to zero. Thus in all, three variables would be equal to zero and remaining two variable *i.e.,* the decision variable x_2 and the slack variable corresponding to the constraint $3x_1 + x_2 \leq 8$ shall be at positive level. Since the number of non-zero variables is less than the number of constraints in the problem, the solution is degenerate.

In such cases, simplex tables can be found repetitive and hence cyclic.

6.9 Self-Assessment Problems

1. A manufacturing unit has three products on their production line. The production capacity for each product is 50, 30 and 45 respectively. The limitation in the production shop is that of 300 manhours as total availability and the manufacturing time required per product is 0.5, 1.5 and 2.0 manhours. The products are priced to result in profits of ₹ 10, 15 and 20 respectively. If the company has a daily demand of 25 units, 20 units and 35 units for respective products, formulate the problem as LP model so as to maximise the total profit.

2. An electronics company produces three types of parts for automatic washing machine. It purchases castings of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines. The selling prices of part A, B and C respectively are ₹ 8, ₹ 10 and ₹ 14. All parts made can be sold. Castings for parts A, B and C respectively cost ₹ 5, ₹ 6 and ₹ 10.

   The shop possesses only one of each type of machine. Costs per hour to run each of the three machines are ₹ 20 for drilling, ₹ 30 for shaping and ₹ 30 for polishing. The capacities (parts per hour) for each part on each machines are shown in the following table.

Machine	*Capacity per hour*		
	Part A	*Part B*	*Part C*
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximise profit for an hour's run. Formulate this problem as an LP model. [Delhi University, M.B.A., 1986]

3. The owner of Metro Sports wishes to determine how many advertisements to place in the selected three monthly magazines A, B and C. His objective is to advertise in such a way that total exposure to principal buyers of extensive sports goods is maximised. Percentage of readers for each magazine are known. Exposure in any particular magazine is the number of advertisements placed multiplied by the number of principal buyers. The following data may be used:

Requirement	*Magazines*		
	A	*B*	*C*
Readers	1 lakh	0.60 lakhs	0.40 lakhs
Principal buyers	15%	15%	7%
Cost per advertisement (₹)	5,000	4,500	4,250

The budgeted amount is at the most ₹1 lakh for advertisements. The owner has already decided that magazine A should have no more than 6 advertisement and B and C each have at least two advertisements. Formulate an LP model for the problem.

[Delhi University, M.B.A., April 1982]

6.10 Summary

The simplex method is used to eradicate the issues in linear programming. The simplex method uses a systematic strategy to generate and test candidate vertex solutions to a linear program. At every iteration, it chooses the variable that can make the biggest modification toward the minimum solution. The simplex method basically takes one by one all the corner points till you reach the optimal one. Simplex basically means a triangle (in 2 dimension) , so graphically, you keep pivoting the corner points till we reach the point of minimum or maximum value. Simplex method, Standard technique in linear programming for solving an optimization problem, typically one involving a function and several constraints expressed as inequalities. The simplex method is a systematic procedure for testing the vertices as possible solutions.

6.11 Key Words/Abbreviations

- **Two phase:** In Two Phase Method, the whole procedure of solving a linear programming problem (LPP) involving artificial variables is divided into two phases.
- **Degeneracy:** Degeneracy in a linear programming problem is said to occur when a basic feasible solution contains a smaller number of non-zero variables than the number of independent constraints when values of some basic variables are zero and the Replacement ratio is same

6.12 Learning Activity

1. What are the limitation of simplex method?

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2. Explain the algorithm of simplex method?

6.13 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. What is the difference between the canonical and the standard form of objective function in simplex application?
2. What condition should be satisfied if the solution obtained through Simplex Algorithm is found to be feasible?

3. What is the necessity to introduce slack and artificial variables from simplex method application?
4. Under what condition, the basic feasible solution is treated as "optimal"?
5. What do you understand by 'Degeneracy' and how does it affect the feasibility of the solution?

B. Multiple Choice/Objective Type Questions

1. Operations Research study generally involves how many phases?

   (a) Three (b) Four

   (c) Five (d) Two

2. In LPP, degeneracy occurs in how many stages?

   (a) One (b) Two

   (c) Three (d) Four

3. The another method to solve a given LPP involving some artificial variable is ____________

   (a) Big M method (b) Method of penalties

   (c) Two-phase simplex method (d) None of the above

4. If there are more than one optimum solution for the decision variable the solution is _________.

   (a) Infeasible (b) Unbounded

   (c) Alternative (d) None of the above

5. A BFS of LPP is said to be ____________ if atleast one of the basic variable is zero.

   (a) Degenerate (b) Non degenerate

   (c) Infeasible (d) Unbounded

Answers

1. (a), 2. (b), 3. (c), 4. (c), 5. (a).

6.14 References

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UNIT 7 DUALITY

Structure:

7.0 Learning Objectives

After studying this unit, you will be able to:

- Define dual problem.
- Describe rules for converting any Primal into its Dual Simplex Method.

7.1 Introduction

Every linear programming problem we discussed so far in the previous chapter has an associated problem called the **dual problem**. The original formulation of the linear programming problem is referred to as the **primal problem**. Each **primal problem** has a comparsion problem which is called the **"dual"**. The dual has the same optimum solution as the primal, but it is derived by an alternative procedure and the analysis of this procedure may be instructive for serveral types of decision problems.

7.2 Duality

The dual contains economic information useful to the management and it may be easier to solve, in terms of the computation than the primal problem. In cases where the primal and the dual problems differ in terms of computation difficulty, we can choose the easier problem (primal or dual) to solve.

Generally if the L.P primal involves maximizing a profit function subject to less-than-or-equal-to resource (physical resource such as hours of machine time or labour time) constraints, the dual will involve minimizing total opportunity costs subject to greater-than-or-equal-to product profit constraints. Formulating a dual problem from a given primal problem is not very complex and once it is formulated, the solution procedure is exactly the same as for any L.P problem.

7.3 Rules for Dual Formulation

1. If the primal is a **maximization** problem, the dual is a **minimization** problem and *vice versa*.
2. **Objective function coefficients** in the primal become right hand side constants of **dual constraints**.
3. The **right hand side constants** in the primal problem are the **coefficients** in the **objective function** of the dual problem.
4. The **row elements** in the primal problem are the **column elements** in the dual problem.

5. **Columns** of constraint coefficients in the primal becomes **rows** of constraints coefficient in the dual.
6. Constraint **inequality** signs are **reversed**.
7. If a primal variable is **non-negative**, the associated dual constraint is a **"greater-than-or-equal-to"** type.
8. If a primal variable is **unrestricted in sign**, the associated dual constraint is **an equation**.
9. If a primal constraint is **an equation**, the associated dual variable is **unrestricted in sign**.
10. If a primal constraint is a **"less-than-or-equal-to"** type, the associated dual variable is **non-negative**.
11. The **dual** of the **dual** is the **primal**.

Illustrations of Primal–Dual Relationship

Illustration 1:

Primal L.P problem:

Maximize profit $Z = 3x_1 - 2x_2 + x_3$

Subject to $2x_1 + 3x_2 - x_3 \leq 6$

$x_1 - 2x_2 + 3x_3 \leq 7$

$2x_1 + 4x_2 + 5x_3 \geq 8$

Or $-2x_1 - 4x_2 - 5x_3 \leq -8$

$x_1 \geq 0 \quad x_2 \geq 0$

x_3 unrestricted in sign

Let y_1, y_2 and y_3 be the variables in the dual problem.

The dual problem is formulated as:

Minimise $C = 6y_1 + 7y_2 - 8y_3$ (Refer rule 1 and rule 2)

$$2y_1 + y_2 - 2y_3 \geq 3$$ (Refer rule 3 and rule 4)

$$3y_1 - 2y_2 - 4y_3 \geq -2$$

$$-y_3 + 3y_2 - 5y_3 = 1$$ (Refer rule 5)

y_1 is unrestricted in sign (Refer rule 6)

$$y_2 \geq 0 \quad y_3 \geq 0.$$

Explanation: Since there are three **'resource'** constraints in the primal, there are three variables (y_1, y_2 and y_3) in the dual. The third primal constraint is multiplied by (–1) to convert it into ≤ type of inequality. *i.e.,* it becomes $-2x_1 - 4x_2 - 5x_3 \leq -8$ which is taken into consideration while writing the dual.

The (–2) in the right-hand-side of the second dual constraint comes from the (–2) coefficient in the primal objective function. The negative right-hand-side constant value is allowable at the stage of formulation and while setting the dual problem for simplex solution, we would multiply that constraint by (–1) or we would use the equivalent version: $-3y_1 + 2y_2 + 4y_3 \leq 2$.

Illustration 2:

Primal problem: Maximize profit $Z = 50x_1 + 120x_2$

Subject to constraints $2x_1 + 4x_2 \leq 80$ (Resource in hours of time of M_1)

$3x_1 + 1x_2 \leq 60$ (Resource in hours of time of M_2)

The dual of this problem has the objective of minimizing the opportunity cost of not using the resources in an optimal manner. Let u_1 and u_2 be the decision variables (when the primal has m constraints (m = 2 in this problem), the 'dual will have m decision variables). u_1 and u_2 represent the worth or opportunity cost of the resource M_1 and M_2 respectively.

The right-hand-side quantities of the primal constraints become the dual's objective function coefficients. The total opportunity cost that is to be minimized will be represented by the objective function:

$$\text{Minimize } C = 80u_1 + 60u_2.$$

The corresponding dual constraints are formed from the transpose of the primal constraints coefficients. Also if the primal constraints are of ≤ type, the dual constraints will be of ≥ type.

The constraints of the dual problem are:

$$\begin{aligned} 2u_1 + 3u_2 &\geq 50 \\ 4u_1 + 1u_2 &\geq 120 \quad \text{Primal profit coefficients} \end{aligned}$$

Coefficients from the second primal constraint

Coefficients from the first primal constraint.

7.4 Significance of the Duality Concept

The concept of duality has two significant purposes. Firstly, the variables in a dual problem **provides important information to the decision makers** in terms of formulating their future plans. Secondly, sometimes the dual problem can be instrumental in obtaining an **optimal solution in lesser number of iterations as** compared to the iterations required to solve the original (primal) problem. For example, an (m × n) primal problem becomes an (n × m) dual problem. If m is very big compared to n, say the primal is of (50 × 3) then the converted dual problem is of (3 × 50) and can be solved quickly, as compared to the primal problem.

Example 2: Convert the given primal problem to a dual problem.

Primal problem: Minimise cost C, when $C = 2x_1 + 5x_2$

Subject to

$$\begin{aligned} x_1 + 3x_2 &\geq 4 \\ 3x_1 + 5x_2 &\geq 7 \\ x_1 + x_2 &\geq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Solution: Since the primal is a minimization problem, the dual will be a maximization problem. Applying the rules for dual formulation, we formulate the dual problem as shown below:

Objective function: Maximise profit Z,

where $Z = 4y_1 + 7y_2 + 3y_3$

subject to $y_1 + 3y_2 + y_3 \leq 2$

$3y_1 + 5y_2 + y_3 \leq 5$

$y_1, y_2, y_3 \geq 0.$

For this primal problem involving only two variables to be solved using simplex tableau the dual offers a distinct computational advantage. Its (dual's) constraints each require only a single slack variable, while each constraint in the primal requires both a surplus and artificial variable to set up the problem for solution by the simplex method.

Example 3: Convert the primal L.P problem given below to a dual problem.

Maximize $Z = 2x_1 + x_2$

Subject to $x_1 + 2x_2 \leq 9$

$x_1 + x_2 \leq 5$

$x_1 - x_2 \leq 2$

$x_1 - 2x_2 \leq 1$

$x_1, x_2 \geq 0.$

Solution: Since the primal is a maximization problem, the dual will be a minimization problem.

Objective function in minimize y,

When $y = 9y_1 + 5y_2 + 2y_3 + y_4$

Subject to constrains, $y_1 + y_2 + y_3 + y_4 \geq 2$

$2y_1 + y_2 - y_3 - y_4 \geq 1$

$y_1, y_2, y_3, y_4 \geq 0.$

Example 4: Write the dual of the following LPP:

Minimize $Z = 10x_1 + 20x_1$ …. (0)

Subject to $3x_1 + 2x_2 \geq 16$ .... (1)

$x_1 + 3x_2 \geq 8$ …. (2)

$2x_1 - x_2 \leq 5$ …. (3)

$x_1, x_2 \geq 0.$

Solution: Since the inequality sign is not same for all three constraint equations, we multiply both sides of equation (3) by (–1). The constraint equation is rewritten as,

$$-2x_1 + x_2 \geq -5$$

Hence the primal is rewritten as,

Minimize $Z = 10x_1 + 20x_2$

Subject to $3x_1 + 2x_2 \geq 16$

$x_1 + 3x_2 \geq 8$

$-2x_1 + x_2 \geq (-5)$

$x_1, x_2 \geq 0.$

The dual is written as shown below:

Maximize $P = 16y_1 + 8y_2 - 5y_3$

Subject to, $3y_1 + y_2 - 2y_3 \leq 10$

$2y_1 + 3y_2 + y_3 \leq 20$

$y_1, y_2, y_3 \geq 0.$

Example 5: Obtain the dual of the LPP given below:

Maximize $Z = 8x_1 + 12x_2 + 6x_3$ …. (0)

Subject to $x_1 - x_3 \leq 5$ …. (1)

$$2x_1 + 4x_2 \leq 12 \qquad \text{.... (2)}$$

$$x_1 + x_2 + x_3 \geq 2 \qquad \text{.... (3)}$$

$$3x_1 + 2x_2 - x_3 = 7 \qquad \text{.... (4)}$$

$$x_1, x_2, x_3 \geq 0.$$

Solution: Since constraint equation 3 is of ≥ type, we have to convert it to ≤ type by multiplying both sides by (–1) to become $-x_1 - x_2 - x_3 \leq -2$.

Constraint 4 is an equality which can be expressed as either

$$3x_1 + 2x_2 - x_3 \leq 7 \quad \text{or} \quad 3x_1 + 2x_2 - x_3 \geq 7$$

Since other constraint equations are of £ type, we express constraint 4 as:

$$3x_1 + 2x_2 - x_3 \leq 7$$

Hence, the primal is expressed in standard form as:

Maximize $Z = 8x_1 + 12x_2 + 6x_3$

subject to

$$x_1 + 0x_2 - x_3 \leq 5$$

$$2x_1 + 4x_2 + 0x_3 \leq 12$$

$$-x_1 - x_2 - x_3 \leq (-2)$$

$$3x_1 + 2x_2 - x_3 \leq 7$$

$$x_1, x_2, x_3 \leq 0$$

The dual is written as shown below:

Minimize $Y = 5y_1 + 12y_2 - 2y_3 + 7y_4$

Subject to

$$y_1 + 2y_2 - y_3 + 3y_4 \geq 8$$

$$4y_2 - y_3 + 2y_4 \geq 12$$

$$-y_1 - y_3 - y_4 \geq 6$$

$$y_1, y_2, y_3 \geq 0$$

Example 6: Obtain the dual of the following primal LPP.

Minimize $Z = x_1 + 12x_2 - 2x_3$

Subject to

$$4x_1 + 2x_2 + 12x_3 \leq 10$$

$$2x_1 - x_2 + 11x_3 \geq (-2)$$

$$x_1 \leq 0 \quad x_2 \text{ unrestricted in sign} \quad x_3 \geq 0.$$

Solution: Let y_1 and y_2 be the dual variables and Y = the objective function.

Objective function is maximize $Y = 10y_1 - 2y_2$

Subject to

$$4y_1 + 2y_2 \geq 1$$

$$2y_1 - y_2 = 12$$

$$12y_1 + 11y_2 \leq (-2)$$

$$y_1 \leq 0, \quad y_2 \geq 0.$$

Relationship between primal and dual problems is given in the *Box 7.1* below:

Box 7.1: Relationship Between Primal and Dual LP Problems

Primal	Will become Dual
(i) Maximization	*(i)* Minimization
(i) Maximization	*(i)* Minimization
(ii) Number of variables	*(ii)* Number of constraints
(iii) Number of constraints	*(iii)* Number of variables
(iv) ≤ type constraint	*(iv)* Non-negative variable
(v) = type constraint	*(v)* Unrestricted variable
(vi) Unrestricted variable	*(vi)* = type constraint
(vii) Objective function coefficient for j^{th} variable	*(vii)* Right hand side constraint for the i^{th} constraint
(viii) Right hand side constant for the j^{th} constraint	*(viii)* Objective function coefficient for 'i'th variable
(ix) Coefficient (a_{ij}) for j^{th} variable in 'i'th constraint	*(ix)* Coefficient (a_{ij}) for i^{th} variable in j^{th} constraint

7.5 Features of the Dual Problem

(i) The coefficients in the objective function of the primal problem are the right hand side constants of the constraint equations in the dual problem.

(ii) The right hand side constants of the constraint equations in the primal problem are the coefficients in the objective function of the dual problem.

(iii) The row elements in the primal problem are the column elements in the dual problem.

(iv) The column elements in the primal problem are the row elements in the dual problem.

(v) The matrix of the dual problem is considered to be the transpose of the matrix of the primal problem.

(vi) The inequalities of "less-than or equal to" (*i.e.,* ≤) in the primal problem are inequalities of "greater than or equal to" (*i.e.,* ≥) type in the dual problem.

7.6 Advantages of Duality

(i) When a primal LPP has a large number of constraints, it is advantageous to solve the dual LPP rather than the primal, because the number of constraints usually equals the number of iterations required to solve the LPP.

(ii) There is no need to add surplus or artificial variables while solving an LPP using simplex method. The problem can be solved quickly using the ***primal-dual method***.

(iii) The dual problem has an important ***economic interpretation***. The dual variables provide an important interpretation of the final solution of an LPP.

(iv) The dual method is useful when investigating changes in the parameters of an LPP (the technique used in referred to as ***sensitivity analysis***).

(v) Duality is used to solve an LPP in which the initial solution is infeasible. The technique used is referred to as *dual* ***simplex method***.

(vi) In some cases, the use of dual helps to ***overcome some computer capacity limitations***.

(vii) Some special procedures developed for ***testing optional situations*** are based on duality.

(viii) The dual is used in developing the MODI algorithm for the transportation model.

7.7 Solved Problems

Problem 7: Given a primal:

Minimise $C = 3x_1 + 4x_2 - 2x_3$

Subject to $1x_1 + 2x_2 - 3x_3 \geq 40$ .... (1)

$3x_1 + 5x_2 + 1x_3 \leq 0$ .... (2)

$0x_1 + 3x_2 + 2x_3 = 30$ .... (3)

Standardise the primal and write its dual.

Solution:

Step 1: Standardise the primal by converting it into maximisation problem by multiplying it by (–1).

We get Maximise $Z = -3x_1 - 4x_2 + 2x_3$

Step 2: Multiply each ³ constraint by (–1) in order to convert it to a £ type. Constraint (1) thus becomes

$$-1x_1 - 2x_2 + 3x_3 \leq -40$$

Step 3: Split each equality constraint into two inequalities. One £ and the other ≥.

Thus constraint (3) is expressed as

$0x_1 + 3x_2 + 2x_3 \leq 30$ .... (3a)

and $0x_1 + 3x_2 + 2x_3 \geq 30$ .... (3b)

Change equation 3(b) to £ type by multiplying by (–1). That is

$-3x_2 - 2x_3 \leq -30$ .... (3b)

The standardised primal is written as

$$\text{Maximise} \quad Z = -3x_1 - 4x_2 + 2x_3$$

subject to

$$-1x_1 - 2x_2 + 3x_1 \leq -40 \quad \text{.... (1)}$$

$$3x_1 + 5x_2 + 1x_3 \leq 80 \quad \text{.... (2)}$$

$$0x_1 + 3x_2 + 2x_3 \leq 30 \quad \text{.... (3)}$$

$$0x_1 - 3x_2 - 2x_3 \leq -30 \quad \text{.... (4)}$$

$$x_1, x_2, x_3 \geq 0$$

The dual is

$$\text{Minimise} \quad C = -40u_1 + 80u_2 + 30u_3 - 30u_4$$

subject to

$$-1u_1 + 3u_2 + 0u_3 - 0u_4 \geq -3$$

$$-2u_1 + 5u_2 + 3u_3 - 3u_4 \geq -4$$

$$3u_1 + 1u_2 + 2u_3 - 2u_4 \geq 2$$

If $u_5 = (u_3 - u_4)$, the objective function is

$$\text{Minimise} \quad C = -40u_1 + 80u_2 + 30u_5$$

Subject to constraints,

$$-1u_1 + 3u_2 \geq -3 \quad \text{.... (1)}$$

$$-2u_1 + 5u_2 + 3u_5 \geq -4 \quad \text{.... (2)}$$

$$3u_1 + 1u_2 + 2u_5 \geq 2, \ u_1, u_2 \geq 0, u_3 \text{ is unrestricted in sign.}$$

Problem 8: *Given a primal*

$$\text{Minimise} \quad Z = 2x_1 + 3x_2 - 1x_3$$

Subject to

$$5x_1 + 1x_2 + 1x_3 \geq 20 \quad \text{.... (1)}$$

$$2x_1 + 1x_2 + 3x_3 = 24 \quad \text{.... (2)}$$

$$1x_1 + 2x_2 - 1x_3 \text{ £ } 18 \quad \text{.... (3)}$$

Write the dual of the above primal.

Solution:

Step 1: Write the primal in the standardised form, constraints (2) and (3) are not in the required form.

Split constraint (2) into two constraints.

$$2x_1 + 1x_2 + 3x_3 \leq 24 \qquad \text{.... (2a)}$$

$$2x_1 + 1x_2 + 3x_3 \geq 24 \qquad \text{.... (2b)}$$

Multiply equation (2a) by (–1) to reverse the direction of inequality from ≤ to ≥.

i.e., $$-2x_1 - 1x_2 - 3x_3 \geq -24 \qquad \text{.... (2c)}$$

Reverse the inequality of constraint (3) from ≤ to ≥ by multiply it by (–1).

i.e., $$-1x_1 - 2x_2 + 1x_3 \geq -18 \qquad \text{.... (3a)}$$

Now the standardised form of primal is

$$\text{Minimise} \quad Z = 2x_1 + 3x_2 - 1x_3$$

Subject to

$$5x_1 + 1x_2 + 1x_3 \geq 20 \qquad \text{.... (1)}$$

$$-2x_1 - 1x_2 - 3x_3 \geq (-24) \qquad \text{.... (2a)}$$

$$2x_1 + 1x_2 + 3x_3 \geq 24 \qquad \text{.... (2b)}$$

$$-1x_1 - 2x_2 + 1x_3 \geq (-18) \qquad \text{.... (3)}$$

Step 2: Write the dual of the standardised primal using the rules for dual formation: The dual is

$$\text{Maximise} \quad W = 20u_1 - 24u_2 + 24u_3 - 18u_4$$

subject to

$$5u_1 - 2u_2 + 2u_3 - 1u_4 \leq 2 \qquad \text{.... (1)}$$

$$1u_1 - 1u_2 + 1u_3 - 2u_4 \leq 3 \qquad \text{.... (2)}$$

$$1u_1 - 3u_2 + 3u_3 + 1u_4 \leq -1 \qquad \text{.... (3)}$$

If $u_5 = (u_2 - u_3)$, then objective function is

$$\text{Maximise} \quad W = 20u_1 - 18u_4 - 24u_5$$

subject to constraints:

$$5u_1 - 1u_4 - 2u_5 \le 2 \quad \text{.... (1)}$$

$$1u_1 - 2u_4 - 1u_5 \le 3 \quad \text{.... (2)}$$

$$1u_1 + 1u_4 - 3u_5 \le -1 \quad \text{.... (3)}$$

or

$$-1u_1 - 1u_4 + 3u_5 \ge 1 \quad \text{.... (3a)}$$

$$u_1,\ u_4 \ge 0,\ u_5 \text{ unrestricted in sign.}$$

Problem 9: *Write the dual of the following primal*

$$\text{Maximise} \quad Z = 50x_1 + 40x_2$$

subject to

$$3x_1 + 5x_2 \le 150 \quad \text{.... (1)}$$

$$1x_2 \le 20 \quad \text{.... (2)}$$

$$8x_1 + 5x_2 \le 300 \quad \text{.... (3)}$$

$$x_1,\ x_2 \ge 0$$

Solution: The problem is written in the standardised form as below:

$$\text{Maximise} \quad Z = 50x_1 + 40x_2$$

$$3x_1 + 5x_2 \le 150 \quad \text{.... (1)}$$

$$0x_1 + 1x_2 \le 20 \quad \text{.... (2)}$$

$$8x_1 + 5x_2 \le 300 \quad \text{.... (3)}$$

$$x_1, x_2 \ge 0$$

The dual is,

$$\text{Maximise} \quad W = 150u_1 + 20u_2 + 300u_3$$

subject to

$$3u_1 + 0u_2 + 8u_3 \ge 50 \quad \text{.... (1)}$$

$$5u_1 + 1u_2 + 5u_3 \ge 40 \quad \text{.... (2)}$$

$$u_1, u_2, u_3 \ge 0.$$

Problem 10: *Obtain the dual of the following primal LPP.*

$$\text{Minimise} \quad Z = 25x_1 + 30x_2$$

subject to

$$5x_1 + 3x_2 \geq 20$$

$$x_1 + 3x_2 \geq 15$$

$$2x_1 - x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Solution: Since it is a minimisation problem, all the constraints should have ³ sign.

Constraint 3 has ≤ sign which can be changed to ≥ sign by multiplying by (–1) both sides of the constraint.

i.e.,

$$-2x_1 + x_2 \geq -10 \qquad \text{.... (3a)}$$

The standardised form of the primal is

$$\text{Minimise} \quad Z = 25x_1 + 30x_2$$

subject to

$$5x_1 + 3x_2 \geq 20 \qquad \text{.... (1)}$$

$$x_1 + 3x_2 \geq 15 \qquad \text{.... (2)}$$

$$-2x_1 + x_2 \geq (-10) \qquad \text{.... (3)}$$

$$x_1, x_2 \geq 0$$

The dual form is as below:

$$\text{Maximise} \quad G = 20y_1 + 15y_2 - 10y_3$$

subject to

$$5y_1 + y_2 - 2y_3 \leq 25$$

$$3y_1 + 3y_2 + y_3 \leq 30$$

$$y_1, y_2, y_3 \geq 0$$

Problem 11: *Obtain the dual of the following LPP:*

$$\text{Minimise} \quad Z = 20x_1 + 10x_2$$

subject to

$$2x_1 + 3x_2 \geq 18$$

$$3x_1 + x_2 \geq 8$$

$$2x_1 - x_2 \leq 6$$

$$x_1, x_2 \geq 0$$

Solution: First the LPP is rewritten in the standardised form with all constraints having ³ sign as the problem has a minimisation objective function.

Constraint number 3 having £ sign is multiplied by (–1) on both sides to obtain the constraint as $-2x_1 + x_2 \geq -6$.

Now the primal is written in the standard form as below:

$$\text{Minimise} \quad Z = 20x_1 + 10x_2$$

subject to

$$2x_1 + 3x_2 \geq 18 \quad \text{.... (1)}$$

$$3x_1 + x_2 \geq 8 \quad \text{.... (2)}$$

$$-2x_1 + x_2 \geq (-6) \quad \text{.... (3)}$$

$$x_1, x_2 \geq 0$$

The dual is written as shown below:

$$\text{Maximise} \quad G = 18y_1 + 8y_2 - 6y_3$$

subject to,

$$2y_1 + 3y_2 - 2y_3 \leq 20 \quad \text{.... (1)}$$

$$3y_1 + y_2 + y_3 \leq 10 \quad \text{.... (2)}$$

$$y_1, y_2, y_3 \geq 0$$

7.8 Self Assessment Problems

1. Write the dual form for the given primal LPP

Maximise $Z = 3x_1 - 2x_2 + x_3$

Subject to $2x_1 + 3x_2 - x_3 = 6$

$x_1 - 2x_2 + 3x_3 \leq 7$

$2x_1 + 3x_2 + 5x_3 \geq 8$

$x_1 \geq 0 \quad x_2 \geq 0$

x_3 unrestricted in sign.

2. Given a primal

Minimise $Y = 3x_1 + 4x_2 - 2x_3$

Subject to $1x_1 + 2x_2 - 3x_3 \geq 40$

$3x_1 + 5x_2 + 1x_3 \leq 80$

$3x_2 + 2x_3 = 30$

Formulate the dual LPP.

3. Given a primal

Maximise $Z = 2x_1 + 3x_2 - 1x_3$

Subject to $1x_1 + x_2 + x_3 \geq 20$

$2x_1 + x_2 + 3x_3 = 24$

$x_1 + 2x_2 - x_3 \leq 18$

Write the dual form of the LPP.

4. Write the dual form of the following primal LPP.

Minimise $Z = 6x_1 + 5x_2$

Subject to $4x_1 + 8x_2 \geq 80$

$$6x_1 + 4x_2 \geq 100$$

$$5x_1 + 5x_2 \geq 95$$

$$6x_1 + 3x_2 \geq 110$$

5. Write the dual of the following primal LPP.

Minimise $C = 6x_1 - 2x_2$

Subject to $3x_1 + 4x_2 \geq 50$

$$x_1 + 2x_2 = 20$$

$$2x_1 + 3x_2 \leq 45$$

6. Write the following primal problem in canonical form and find its dual.

Maximise $Z = 3x_1 + x_2 + 5x_3 + 3x_4$

Subject to $3x_1 + x_2 + 2x_3 = 30$

$$2x_1 + x_2 + 3x_3 + x_4 \geq 15$$

$$2x_2 + 3x_3 \leq 25$$

$$x_1, x_2, x_3, x_4 \geq 0$$

7. Formulate the dual of the following primal problem:

Maximize $Z = 100x_1 + 70x_2 + 30x_3$

Subject to, $20x_1 + 10x_2 + 5x_3 \leq 1000$

$$x_1 + x_2 + x_3 \leq 50$$

$$x_1, x_2, x_3 \geq 0$$

8. Formulate the dual of the following primal problem.

Maximize Y, where $Y = 6y_1 + 2y_2 + 5y_3$

subject to $y_1 + y_2 + y_3 \leq 400$

$$2y_1 + 3y_2 - 4y_3 \geq 800$$

$$y_1 - 10y_2 + 20y_3 = 200$$

$y_1 \geq 0$, y_2 unrestricted in sign $y_3 \geq 0$.

9. Formulate the dual of the following primal problem.

Minimize $Z = 2x_1 + 6x_2 - 4x_3$

Subject to $x_1 + x_2 + x_3 \leq 300$

$$2x_1 - 2x_2 + 7x_3 \geq 10$$

$$x_1 - x_2 + 3x_3 = 50$$

$$x_1 + x_2 \geq 20$$

$x_1\ ^3\ 0\ x_2 \geq 0\ x_3$ unrestricted in sign

10. Formulate the dual of the LPP given below:

Maximise Profit $Z = 80x_1 + 75x_2$

subject to $x_1 + 3x_2 \leq 4$

$$2x_1 + 5x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Find the dual of the problem's dual.

11. Write the dual of the following LP problem.

Primal: Minimise cost $= 120x_1 + 250x_2$

subject to $12x_1 + 20x_2 \geq 50$

$$x_1 + 3x_2 \geq 4$$

7.9 Summary

The Duality in Linear Programming states that every linear programming problem has another linear programming problem related to it and thus can be derived from it. The original linear programming problem is called “Primal,” while the derived linear problem is called “Dual.

7.10 Key Words/Abbreviations

- **Primal:** In the primal problem, the objective function is a linear combination of n variables. There are m constraints, each of which places an upper bound on a linear combination of the n variables.
- **Dual:** Each variable in the primal LP becomes a constraint in the dual LP.

7.11 Learning Activity

1. According to you, how to explain duality in economics.

   ____

   ____

7.12 Unit End Questions (MCQ and Descriptive)

A. Descriptive Type Questions

1. What is “duality”? Distinguish between a ‘primal problem’ and a dual problem in linear programming.
2. State the rules for formulation of a dual problem.
3. Discuss the relation between a primal problem and its dual with an illustration.
4. Discus the significance of the dual concept.
5. State the features of the dual problem.

B. Multiple Choice Questions

1. If a primal variable is unrestricted in sign, the associated dual constraint is __________.

   (a) an inequation

   (b) complex variable

   (c) an equation

   (d) none of these

2. If a primal constraintis a less than or equal to type, the associated dual variableis ______.

   (a) non-negative.

   (b) Positive

   (c) less than type

   (d) equal to type

3. The dual of the dual is the ___________________.

   (a) Dual

   (b) primal-dual

   (b) dual-primal

   (d) primal

4. If there are three resource constraints in the primal, then the number of variables in the dual are ______________________.

   (a) Two

   (b) Three

   (c) One

   (d) Four

5. Objective function coefficient for j^{th} variable in the primal function becomes_____________

   (a) Objective function coefficient for 'i'^{th} variable in the dual

   (b) Coefficient (a_{ij}) for i^{th} variable in j^{th} constraint in the dual

   (c) Right hand side constraint for the i^{th} constraint in the dual

   (d) None of these

6. The matrix of the dual problem is considered to be ___________________.

   (a) The transpose of the matrix of the primal problem.

   (b) The adjoint of the matrix of the primal problem.

(c) The transpose of the matrix of the dual problem.

(d) the adjoint of the matrix of the primal problem.

7. Duality is used to solve an LPP in which the initial solution is infeasible. The technique used is referred to as ________________________.

(a) Simplex method (b) Big M method

(c) Two phase simplex method (d) Dual simplex method

Answers:

1. (c), 2. (a), 3. (d), 4. (b), 5. (c), 6. (a), 7. (d).

7.13 References

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UNIT 8 TRANSPORTATION PROBLEMS

Structure:

8.0 Learning Objectives

After studying this unit, you will be able to:

- Discuss the concept of formation of loops in transportation table.
- Explain various steps for the initial solution of the problem through various methods, namely North-West Corner Rule least Cost Method and Vogel's Approximation Method.
- Describe the flow chart for solution through computer application.
- Illustrate the steps for optimal solution through optimality test, namely MODI method and stepping stone method.
- Explain the concept of Degeneracy and its solution.
- Discuss the methodology through the application of various methods under different condition (through solved examples).
- Analyse yourself through self assessment problems.
- Define transportation problem
- Describe stream line simplex method for the transportation problem

8.1 Introduction

All linear programming problems can be solved by simplex method, but certain special problems lend themselves to easy solution by other methods. One such case in that of Transportation problems.

Transportation problems are encountered in physical distribution of goods. Source of supply, availability of material or commodity for distribution, the requirement of demand at particular place or destination or at number of destinations are some of the parameters involved in the problem. The objective is to minimise the cost associated with such transportation from place of supply to places of demand within given constraints of availability and level of demand. These distribution problems are amenable to solution by a special type of LP model know as 'Transportation Model'. It can also be applied to the maximisation of some utility value such as financial resources.

8.2 Problem Statement

Let a_i = quantity of product available at origin i

b_j = quantity of product required at destination j

c_{ij} = cost of transporting one unit of product from origin i to destination j

x_{ij} = quantity transported from origin i to destination j

Assume that $\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_j$

It is the case when demand is fully met from the origin. The problem can be stated as LP problem in the following manner.

Min (total cost) $Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\ x_{ij}$

Subject to $\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1, 2, 3..............\ m$

$\sum_{i=1}^{m} x_{ij} = b_j$ for $j = 1, 2, 3.................\ n$

and $x_{ij} \geq 0$ for all $i = 1, 2, 3..............m$

$j = 1, 2, 3.............\ n$

This can be represented as a matrix within a matrix of the dimensions mxn. One matrix is the unit cost matrix which represents the unit transportation cost for each of the possible transportation routes. Superimposed on this matrix is the matrix in which each cell contains a transportation variable, *i.e.,* the number of units shipped from the row-designated origin to the column designated destination. The amount of supplies *ai* available at source *i* and amount demanded *bj* at each destination *j i.e., ai*'s and *bj*'s represent supply and demand constraint. The problem can be solved either by simplex method already explained in the previous chapters or by transportation method.

8.3 Some useful Requirements

1. In the transportation model, the two parameters *i.e.,* supply and demand have some cumulative total. Thus, it can be said that the material available for supply can be supplied because the demand exists at the same level. It is a case of balanced transportation problem.

In actual life situation, the demand may exceed the supply available or *vice versa*. It is termed as an unbalanced transportation problem.

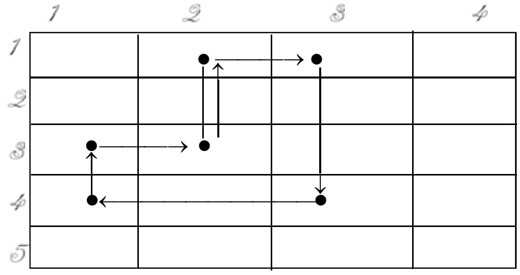
2. When the number of positive allocations in the feasible solution is less than (rows + columns -1), the solution is said to be degenerate. For feasibility criterion, $m + n - 1$ = number of allocations (m = number of rows, n = number of columns in the matrix).
3. Wherever there is a positive allocation to a particular transportation cell, it is called an occupied cell. Other cells of the matrix are treated as empty or unoccupied cells.

8.4 Loops in the Transportation Table

Since any basic feasible solution must contain ($m + n - 1$) independent non-zero allocations, where $m \times n$ is the size of the transportation matrix *i.e.,* row × column numbers, independent non-zero allocations imply that we cannot form a closed circuit (loop) by joining positive allocations by horizontal and vertical lines only. Hence, for the formation of a loop, following conditions must satisfy.

1. Any two adjacent cells of the ordered set lie either in the same row or in the same column.
2. No three or more adjacent cells in the ordered set lie in the same row or the column. The first cell of the set must be the last in the set.

To illustrate the above conditions, let us consider the following table.



The ordered set of cells contain the following allocated cells, (1, 1), (1, 2), (2, 1), (2, 2), (5, 2), (2, 4), (5, 4). The loop formation is for cells (1, 1) (1, 2) (2, 1) and (2, 2) and cells (2, 2), (2, 4), (5, 4) and (5, 2) as (2, 2) appears twice. Whereas the loop formation in the following table satisfies all the conditions.

The loop (1, 2) (1, 3), (3, 4), (4, 1), (3, 1) (3, 2) and (1, 2) is feasible loop satisfying all the conditions of loop formation.

Thus, we can say that

1. Every loop has an even number of cells and the least being four.
2. The allocations are in independent position, if it is not possible to reduce or increase the independent individual allocation without altering the position of allocation.
3. Each row and column in the matrix should have only one plus and minus sign. The loop must start with an empty cell and all other cells forming the loop must be occupied or allocated cells.
4. Closed loop may or may not be rectangular in shape.

8.5 Steps in Transportation Methods

The solution of the transportation problem has the following algorithm

Step 1. Formulate the problem and establish the transportation matrix or table, the cells indicating the parameters value for various combinations *i.e.,* cost, profit, time, distance etc.

Step 2. Obtain an initial basic solution. This can be done in three different ways *i.e.,* North-West Corner Rule, Least Cost Method or the Vogel's Approximation Method.

The initial basic solution from any of the methods named above should satisfy the following conditions.

(*i*) The solution must be feasible, satisfying allocation all supply requirement into demand position.

(*ii*) The number of positive allocations must be equal to $m + n - 1$, otherwise the solution will become degenerate.

Step 3. Test the initial solution for optimality—This is done either by Stepping Stone Method or by MODI Method.

Step 4. Update the solution *i.e.,* applying step 3 till optimal feasible solution is obtained.

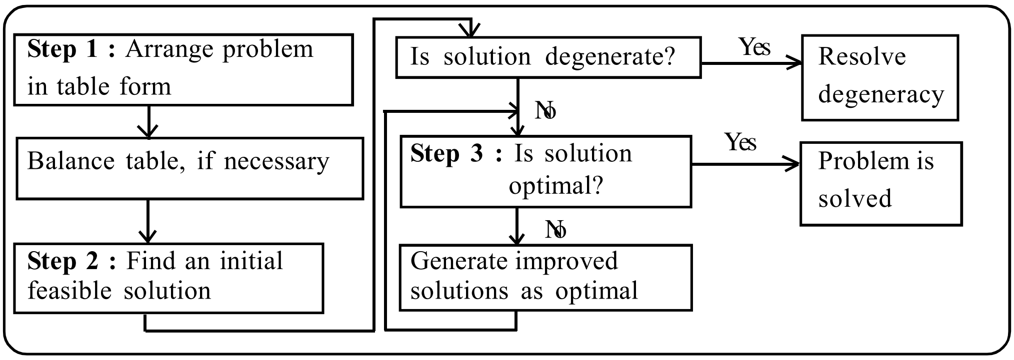
8.6 Transportation Matrix or Table

The illustration of the transportation model can best be represented by taking an example. The matrix is written as follows.

To / *From*	*D*	*E*	*F*	*Supply*
A	6	4	1	50
B	3	8	7	40
C	4	4	2	60
Demand	20	95	35	150

A, B, C are sources of supply and *D, E, F* the destinations of demand. The matrix indicates the cost of transportation per unit item from source *A, B, C* to the destination *D, E, F*.

8.7 Schematic diagram of Solving Transportation Problems for Computer Application



8.8 Methods of Solving Transportation Problem

Following methods can be used for solving transportation problem:

1. North-West Corner Rule (N.W. Corner Rule) or DENTZY's Method.
2. Least Cost Method (LCM)
3. Vogel's Approximation Method (VAM).

8.8.1 North-West Corner Rule

Initial basic feasible solution can be obtained as follow:

(*a*) If $a_1 > b_1$, assign b_1 in the cell in the first column of the first row. Then put $x_{11} = bi$ and proceed horizontally to the next column in the first row until the supply of this origin is exhausted.

(*b*) If $a_1 < b_1$, assign the value equal to a_1 as the value of x_{11}, and then proceed vertically below to the next row until the demand of this destination is satisfied.

(*c*) If $a_1 = b_1$, then put the value of x_{11} equal to a_1 or b_1 and then proceed diagonally to the cell determined by the next column of the next row.

In this way, move horizontally until a supply source is exhausted, vertically down until a destination demand is satisfied and diagonally, when the demand at the destination matches exactly the supply available, until the South-East Corner is reached.

8.8.2 Least Cost Method (LCM)

The NW corner rule given above considers only the availability and supply requirement in making assignments, without giving any thought to the involvement of cost. It is, therefore, not a sound solution, as it ignores the most important factor 'Cost' which is to be determined or optimised.

The Least Cost Method can be applied in the following way:

Step 1. Select the lowest cost cell in the whole matrix *i.e.* out of all values of rows and columns of the transportation table. In case of a tie, select arbitrarily.

Step 2. Allocate maximum possible units considering the supply as well as demand values to this selected lowest cost cell.

Step 3. Eliminate the row or column satisfied fully by the above allocation, if feasible.

Step 4. Adjust the capacity and requirement (supply/demand) for the remaining values after the above allocation.

Step 5. Repeat Step 1 to 4 for the reduced cost table until all the capacities and requirements are fully satisfied.

8.8.3 Vogel's Approximation Method (VAM)

This is a preferred method over other two methods due to its solution being either optimal or very near optimal. This may reduce the time for optimal calculations.

1. Consider each row of the cost matrix individually and find the difference between two least cost cells in it. Then repeat the exercise for each of the columns. Identify the column or row with the largest value difference. In case of tie, select any one (it is wise to select the row or column to give allocation at minimum cost cell). Now consider the cell with the minimum cost in the column (or row, as the case may be) and assign the maximum units possible, considering the demand and supply positions corresponding to that cell. Assign only one cell at a time.
2. Delete the column/row, which has been satisfied.
3. Again, find out the differences of least cost cells and proceed in the same way. Continue until all units have been assigned.

The Vogel's Approximation Method is also called the Penalty Method because the cost differences that it uses are nothing but the penalties of not using the least cost route. Since the objective function is the minimisation of the transportation cost, in each iteration that route is selected which involves the maximum penalty of not being used.

Initial feasible solution for the earlier example is obtained in Problem 21.3.

8.9 Testing the Optimality

Having obtained the initial basic feasible solution by any of the three methods described above, we have to test the solution, if we have reached the optimal level. We can do this by two methods.

1. Stepping Stone Method.
2. Modified Distribution (MODI) Method.

8.9.1 Stepping Stone Method

By using stepping stone method, we calculate the opportunity cost of each empty cell. We find out as to what effect on the total cost would be if one unit is assigned to any empty cell. The total cost level would indicate if it is more than that obtained by initial feasible solution. If cost is reduced, solution is not optimal. If it does not, then we have reached optimal solution.

Things to Remember

1. In the stepping stone method, the occupied cells or the circled members are called *stones* and the cells containing these circled numbers are called *stone cells.* The unoccupied cells are called *water cells.*
2. The cells used for re-allocation are given plus and minus signs. Wherever we wish to increase the allocation, it is given plus sign, and when we want to reduce allocation, it is given minus sign. This would mean increase or reduction of transportation costs.
3. Closed loop starts with the unoccupied cell whose additional allocation is being tested, but has to have minimum of three occupied cells to work out the optimality. Horizontal and vertical moves are made in clockwise direction through these occupied cells only. This is primarily to ensure that any increase in a row/column must be compensated by equivalent reduction to balance the supply/demand or capacity requirement conditions.

8.9.2 MODI Method

The modified distributions method (MODI method) can also be used for testing the optimality of the solution obtained for a transportation problem. This is called U – V method also. By this method, the solution can be gradually improved heading towards the optimal value.

Following steps are to be followed to apply this method for the optimality test of the problem.

Step 1. For a given solution of the transportation problem in the form of allocated and unallocated cell matrix, we calculate auxiliary variables the Ui for $i = 1, 2, 3 ....m$ and Vj for $j = 1, 2.........n$. for rows and column respectively. The values of Ui and Vj are calculated by using the relationship $Cij = Ui + Vj$ for all i, j for all occupied cells. To start, Ui or Vj can be selected as zero arbitrarily for the allocations in row/column.

Step 2. For unallocated or unoccupied cells, Δij can be calculated by the relationship $\Delta ij = Cij - (Ui + Vj)$

where Δij is called cell evaluation index or the opportunity index.

Step 3. If $\Delta ij > 0$ then optimal solution has been reached.

If $\Delta ij = 0$, the solution remains unchanged and an alternate solution is feasible.

If $\Delta ij < 0$, there can be an improved solution by introducing cell (i, j) in the basis.

Step 4. We select an unallocated cell with maximum negative opportunity cost of all unallocated cells.

Step 5. Follow a closed path for the unoccupied/unallocated cells obtained in step 4 and assign + (plus) and – (minus) alternately starting with plus for the selected unallocated cell.

Step 6. Now assign largest units possible to the unallocated cell satisfying problem conditions, the smallest allocation in a cell with the minus sign on the closed path will indicate the number of units that can be shifted to the unallocated cell. This quantity is added to all the allocation cell on the closed path marked with plus sign and subtracted from those allocated cells with minus sign.

Step 7. Calculate cost of transportation from the modified allocations and repeat the process through steps 1 to 7 till we reach all the values of $\Delta ij \geq 0$. This would indicate the optimal solution of the problem. The transportation cost (optimal) can now be calculated with this modification.

The above mentioned procedure can best be explained by its application to an actual problem and obtaining the optimal cost by the iteration process described above.

8.10 Degeneracy

For a feasible transportation optimal solution, there should be $m + n - 1$ occupied cells or allocations, whenever the number of occupied cells is less than $m + n - 1$, the solution is called "degenerate" and it cannot be tested for optimality. Therefore, a special procedure need to be followed as under.

Degeneracy in the initial feasible solution — In this case, we allocate Î (every small amount) to the empty cell of the solution, to bring the allocation to the desired level (*i.e.,* $m + n - 1$). It is to be done to the least cost empty cell in minimisation problem. The problem is then solved as if it were non-degenerate. Optimality check can now be conducted. If this assignment of Î to the least cost cell is not lending the problem for optimality test, then Î to be assigned to second lowest cell instead and so on.

Degeneracy in the intermediate solution—In this case, ∈ is assigned to one or more of the newly vacated cells. Having brought the solution to $m + n - 1$ occupied cells level, optimality test can be carried out.

As an example, following problem can be considered.

	C	D	Supply
A	3	3	50
B	4	6	30
Demand	50	30	

⟶

Initial degenerate solution

A	3(50)	3
B	4	6(30)

⟶

	C	D
A	3(50)	3(∈)
B	4	6(30)

⟶

	C	D
A	3(20)	3(30)
B	4(30)	6

Now instead of 2, there are 3 allocations = 2 + 2 – 1, hence feasible solution.

8.11 Stream Line Simplex Method for the Transportation Problem

The transportation simplex method uses linear programming to solve transportation problems. The goal is to create the optimal solution when there are multiple suppliers and multiple destinations. ... We create a transportation matrix with this data and apply one of the many methods to find a basic feasible solution (BFS).

We will solve this problem using the streamlined Simplex algorithm for transportation problems. In the first phase, we will apply the Vogel's method to construct an initial basic feasible solution; and in the second phase, where the task is to iterate toward an optimal solution, we will apply the u-v method to conduct optimality tests.

8.12 Solved Examples

Problem 1:

Solve the problem given in para 21.7 by North-West Corner Rule.

Solution:

By this method, the assignments, for the matrix given under para 21.7 above, are obtained as given in the following table. Check for the balanced supply/demand situation. It is satisfied.

To / *From*	*D*	*E*	*F*	*Supply*
A	6 20	4 $\boxed{30}$	1	50
B	3	8 $\boxed{40}$	7	40
C	4	4 $\boxed{25}$	2 $\boxed{35}$	60
Demand	20	95	35	150

The total cost of transportation = 6 × 20 + 4 × 30 + 8 × 40 + 4 × 25 + 2 × 35 = 730.

Here initial demand of 20 for destination *D* has been met with from *A*, starting from NW corner. Since supply for *A* can be 50, we move horizontally and allocate remaining 30 to destination *E*. This satisfies demand for *D* and total supply from *A*. Now we move vertically down to meet further demand of *E* (total 95, remaining 95 –30 = 65). Since *B* can supply only 40 it is allocated to cell *BE*. Remaining demand of *E i.e.,* 65 – 40 = 25 has to be allocated from *C*, which can supply 60 units. 25 are allotted to *E* and the remaining 35 to *F* as indicated on the matrix. Now we have reached SE corner of the matrix satisfying all supply/demand requirements.

Feasibility of the Solution—After all the allocations, we have to check for feasibility. For this, we should have $m + n - 1$ occupied cells. Then we have a feasible solution. In the above case, there are 5 allocations which is $m + n - 1 = 3 + 3 - 1 = 5$. Hence the solution obtained is feasible one.

Problem 2:

Solve the following problem by Least Cost Method.

To / *From*	*D*	*E*	*F*	*Supply*
A	6	4	1	50
B	3	8	7	40
C	4	4	2	60
Demand	20	95	35	150

Solution:

Check for the balanced problem *i.e.* whether demand = supply. Here it is a balanced problem as both the sides add to 150.

Step 1. Minimum Cost in the matrix is 1 at cell *AF*.

Step 2. We allocate 35 units to cell *AF*, being the maximum of supply and demand for cell *AF*.

	D	*E*	*F*		*Supply*
A	6	4	1	35	50
B	3	8	7		40
C	4	4	2		60
Demand	20	95	35		

Step 3. Since Demand for *F* has been satisfied, the column *F* has been eliminated.

Step 4. Adjusted cost matrix will become

	D	*E*	*Supply*
A	6	4	(50 - 35) = 15
B	3	8	40
C	4	4	60
Demand	20	95	

Step 5. Repeating Steps 1 to 4, we get the allocations as

	D		*E*		*F*	
A	6		4	15	1	35
B	3	20	8	20	7	
C	4		4	60	2	

Calculating the cost as per these allocations, we get

Total cost of transportation $= 3 \times 20 + 4 \times 15 + 8 \times 20 + 4 \times 60 + 1 \times 35$

$= 60 + 60 + 160 + 240 + 35 = 555$

This value of transportation cost is lesser than the cost obtained by the North-West Corner Rule. Thus, this is an improved method and should be used as initial solution for optimality test.

Feasibility Checking

If allocations are $m + n - 1$, than the solution is feasible.

In this case, $m + n - 1 = 5$

and we have 5 allocations or occupied cells.

Hence, the solution obtained is feasible.

Problem 3:

Solve the Problem given in para 4.7 by VAM.

Solution:

To / *From*	*D*	*E*	*F*	*Supply*	*Iterations*	
					I	*II*
A	6	4 $\boxed{15}$	1 $\boxed{35}$	50	3	3
B	3 20	8 $\boxed{20}$	7	40	4	1
C	4	4 $\boxed{60}$	2	60	2	2
Demand	20	95	35	150		
Iterations I	I	0	1			
II	—	0	1			

Row A has minimum element as 1 and next least as 4, the difference 4–1 = 3 is written against iteration I in row A, similarly for row B, the difference of least cells will be 7–3 = 4 and is so indicated under iteration I, row B. The process is repeated for C row and all the columns. Maximum of Values in row and column differences, *i.e.,* 3, 4, 2, 1, 0, 1 is 4 and hence allocation (max. = 20) is made to the cell of least cost *i.e.,* cell BD. This satisfies column D and is scored out. Repeating same process during Iteration II, we allocate 35 units at Cell AF (cell with least cost in row A). This satisfies column F fully and hence Column F is scored out, due to its demand having been met fully. Other allocations are made based on supply/demand positions.

It can be seen that total cost involved in VAM comes to 555, a solution better than the one obtained by NW corner rule involving a total cost of 730, but similar to that obtained by the Least

Cost Method. Solution is also feasible as there are $m + n - 1 = 3 + 3 - 1 = 5$ occupied cells in the above solution.

Problem 4:

Test the optimality by Stepping Stone Method.

Solution:

Applying Stepping Stone Method, let us consider allocation to the unoccupied cell AD with closed loop AD → AE → BE → BD (AE, BE and BD are three occupied cells).

Increase in cost by adding 1 unit at AD = +6

Increase in cost by decreasing 1 unit at AE = -4

Increase in cost by increasing 1 unit at BE = +8

Increase in cost by decreasing 1 unit at BD = -3

Net effect = +7

It indicates increase of cost by changing allocation for one unit at cell AD. Hence, solution has not improved.

Let us consider other routes as follows:

Unoccupied Cell	*Closed Loop*	*Net Cost Change*
BF	BF – BE – AE – AF	+ 7 – 8 + 4 – 1 = + 2
CD	CD – BD – BE – CE	+ 4 – 3 + 8 – 4 = + 5
CF	CF – CE – AE – AF	+ 2 – 4 + 4 – 1 = + 1

Since any allocation to any empty cell increases the cost, the solution obtained is optimal.

Similarly, while testing the optimality of the initial basic feasible solution obtained by VAM, we find that solutions are identical and optimal.

Problem 5:

Find the optimal solution for the cost and supply/demand matrix as given below:

Supply Points	*Destinations*				*Supply*
	D_1	*D_2*	*D_3*	*D_4*	
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Demand	5	8	7	15	35

Solution:

This is a balaced Supply/Demand problem. The optimal solution by VAM is given below.

	D_1	*D_2*	*D_3*	*D_4*
P_1	19 5	30	50	12 2
P_2	70	30 3	40 7	60
P_3	40	10 5	60	20 13

Transportation Cost = 19 × 5 + 30 × 3 + 10 × 5 + 40 × 7 + 12 × 2 + 20 × 13 = 799.

Let us now apply the optimality test to unoccupied cells.

Considering Closed loops

for cell (P_1-D_2), loop is $P_1 D_2 \rightarrow P_1 D_4 \rightarrow P_3 D_4 \rightarrow P_3 D_2$

Increase in cost= 30 - 12 + 20 - 10 = 38

for (P_1-D_3), loop is $P_1 D_3 \rightarrow P_1D_4 \rightarrow P_3 D_4 \rightarrow P_3 D_2 \rightarrow P_2 D_2 \rightarrow P_2 D_3$

Cost increase = 50 - 12 + 20 - 10 + 30 - 40 = 38

Similarly for (P_2-D_1), loop is $P_2 D_1 \rightarrow P_1 D_1 \rightarrow P_1D_4 \rightarrow P_3 D_4 \rightarrow P_3 D_2 \rightarrow P_2 D_2$

Cost increase= 70 - 19 + 12 - 20 + 10 - 30 = 23

Since all the values are positive, it is optimality level, all revised allocations resulting in cost increase.

Solution is therefore optimal.

Problem 6:

The distribution of commodity from warehouses *A, B, C* and *D* is planned to three sources *P, Q* and *R*. The level of surpluses and requirements at various sources are given in the following matrix with related cost of transportation as cells of the matrix.

	P	*Q*	*R*	*Surpluses*
A	2	7	4	5
B	3	3	1	8
C	5	4	7	7
D	1	6	2	14
Requirements	7	9	18	㉞

Work out the optimal cost of distribution.

Solution:

It is confirmed that it is a balanced problem (to be checked before start).

Let us obtain the initial feasible solution of the problem for calculating the minimum cost of the transportation, by using Vogel's Approximation Method. The initial solution so obtained is as follows:

	P		*Q*		*R*	
A	2	$\boxed{5}$	7		4	
B	3	$\boxed{2}$	3		1	$\boxed{6}$
C	5		4		7	$\boxed{7}$
D	1		6	$\boxed{9}$	2	$\boxed{5}$

It is a feasible solution having $m + n - 1 = 4 + 3 - 1 = 6$ allocations.

The transportation cost involved in these allocations is

$2 \times 5 + 3 \times 2 + 6 \times 9 + 1 \times 6 + 7 \times 7 + 2 \times 5 = 135$

Optimality Test

For this initial feasible solution, we now proceed to test the optimality of the solution by using MODI method, following step wise iterations.

Step 1. Since there are three allocations (max) in column *R*, we can arbitrarily select $V_R = 0$.

Now based on the occupied cells values and using $Cij = Ui + Vj$, we get the values of *Ui* and *Vj* as follows.

$$C_{DR} = U_D + V_R$$

Hence $U_D = 2 \quad (\text{Since } V_R = 0)$

Similarly $U_C = C_{CR} - V_R = 7 - 0 = 7$

$U_B = C_{BR} - V_R = 1 - 0 = 1$

and $V_P = C_{BP} - U_B = 3 - 1 = 2$

$V_Q = C_{DQ} - U_D = 6 - 2 = 4$

$U_A = C_{AP} - V_P = 2 - 2 = 0$

These can be superimposed on the matrix as follows:

	P	*Q*	*R*	*Ui*
A	2 $\boxed{5}$	7	4	0
B	3 $\boxed{2}$	3	1 $\boxed{6}$	1
C	5	4	7 $\boxed{7}$	7
D	1	6 $\boxed{9}$	2 $\boxed{5}$	2
Vj	2	4	0	

Step 2. Now we calculate opportunity costs for all unoccupied cells by relation

$\Delta ij = Cij - (Ui + Vj)$.

Unoccupied Cells	*Change in Cost (Dij)*
CP	$5 - (7 + 2) = -4$
DP	$1 - (2 + 2) = -3$
AQ	$7 - (0 + 4) = 3$
BQ	$3 - (1 + 4) = -2$
CQ	$4 - (7 + 4) = -7$
AR	$4 - (0 + 0) = 4$

Step 3. Since all the values of D*ij* are not zero or positive, the solution is not optimal. Hence, we proceed to step 4.

Step 4. Maximum negative cost in the above table is for unoccupied cell CQ. Hence, this cell CQ is selected to be brought into the basis.

Step 5. For Cell CQ, we construct closed path as follows:

CQ → CR → DR → DQ → CQ

We assign plus (+) and minus (-) alternately on the closed path starting plus (+) for the selected cell CQ.

Hence, CQ is assigned (+)

CR is assigned (-)

DR is assigned (+)

DQ is assigned (-)

Step 6. Based on the problem conditions of supplies and requirements, we can assign maximum value to CQ as 7 (CQ can be assigned either from CR or from DQ. Hence value 7 is assigned to CQ).

And adjusting relevant supplies/demand positions, the revised allocation will be

	P	*Q*	*R*
A	2 $\boxed{5}$	7	4
B	3 $\boxed{2}$	3	1 $\boxed{6}$
C	5	4 $\boxed{7}$	7
D	1	6 $\boxed{2}$	2 $\boxed{12}$

Step 7. Cost of transportation by new allocations = 2 × 5 + 3 × 2 + 4 × 7 + 6 × 2 + 1 × 6 + 2 × 12 = 86 (an improvement over Initial allocations costing 135).

For optimality, we repeat steps 1 to 7.

Step 1 (R_1). With allocation thus made, we again calculate *Ui* and *Vj*'s and the change in costs for the unoccupied cells.

Arbitrarily, selecting $V_R = 0$, we get

$U_B = 1$

$U_D = 2$

and similarly $U_A = 0$

$U_C = 0$

$V_Q = 4$

$V_P = 2$

Step 2 (R_1). Now let us calculate D*ij* for all unoccupied cells

$D_{AQ} = 7 - (0 + 2) = 5$

$D_{AR} = 4 - (0 + 0) = 4$

$D_{BQ} = 3 - (1 + 4) = -2$

$D_{CP} = 5 - (0 + 2) = 3$

$D_{CR} = 7 - (0 + 4) = 3$

$D_{DP} = 1 - (2 + 2) = -3$

Step 3 (R_1). Since all the values of D*ij* are not zeros or positive, the solution is not optimal and hence we proceed to step 4.

Step 4 (R_1). The cell with maximum negative changed cost is DP and hence it is brought into basis.

Step 5 (R_1). The closed path for unoccupied cell DP is DP → BP → BR → DR → DP and we assign plus and minus signs alternately.

Step 6 (R_1). Looking at the supply requirement conditions, we can assign 2 to cell DP and readjustment is carried out to obtain the modified allocations.

	P		*Q*		*R*	
A	2	$\boxed{5}$	7		4	
B	3		3		1	$\boxed{8}$
C	5		4	7	7	
D	1	$\boxed{2}$	6	2	2	$\boxed{10}$

***Step 7* (R_1).** The transportation cost for these allocations is

$2 \times 5 + 1 \times 2 + 4 \times 7 + 6 \times 2 + 1 \times 8 + 2 \times 10 = 80$

This is an improvement over the last allocation. But we have to test its optimality again.

We repeat step 1 to 7 again.

***Step 1* (R_2).** Calculating the values of *Ui*s and *Vj*s, we get

$U_D = 0$ (arbitrarily) since there are max allocations in this row.

Hence $V_P = 1, V_Q = 6, V_R = 2$

and $U_C = -2, U_B = 1, U_A = 1$

***Step 2* (R_2).** Calculating D*ij* for all unoccupied calls, we get

$$D_{BP} = 3 - (1 + 1) = 1$$

$$D_{CP} = 5 - (-2 + 1) = 6$$

$$D_{AQ} = 7 - (1 + 6) = 0$$

$$D_{BQ} = 3 - (1 + 6) = -4$$

$$D_{AR} = 4 - (1 + 2) = 1$$

$$D_{CR} = 7 - (-2 + 2) = 7$$

***Step 3* (R_2).** Since all the values of D*ij* are not zeros or positive, we proceed to step 4, as the solution is not optimal.

***Step 4* (R_2).** Maximum negative cost is for cell BQ and hence it is brought into basis.

***Step 5* (R_2).** We draw close path for the selected cell BQ. The path is BQ → BR → DR → DQ → BQ and alternate plus and minus signs are assigned.

***Step 6* (*R*$_2$).** We can allocate 2 to this cell BQ as per problem conditions and hence allocations change as follows.

	P		*Q*		*R*	
A	2	$\boxed{5}$	7		4	
B	3		3	$\boxed{2}$	1	$\boxed{6}$
C	5		4	$\boxed{7}$	7	
D	1	$\boxed{2}$	6		2	$\boxed{12}$

***Step 7* (*R*$_2$).** The transportation cost for these allocations is

$2 \times 5 \times 1 \times 2 + 3 \times 2 + 4 \times 7 + 1 \times 6 + 2 \times 12 = 76.$

This is also an improvement over the last allocation. We have to check its optimality once again.

Step 1(R$_3$). Calculating *Ui*, and *Vj*s, we get

$V_R = 0, V_Q = 2, V_P = 1$

$U_B = 1, U_D = 2, U_C = 2, U_A = 1$

Step 2 (R$_3$). Calculating D*ij* for all unoccupied cells, we get

$D_{BP} = 3 - (1 + 1) = 1$

$D_{CP} = 5 - (2 + 1) = 2$

$D_{AQ} = 7 - (1 + 2) = 4$

$D_{DQ} = 6 - (2 + 2) = 2$

$D_{AR} = 4 - (1 + 0) = 3$

$D_{CR} = 7 - (2 + 0) = 5$

Step 3 (R$_3$). Since all the values of D*ij* *i.e*., change in costs for all the unoccupied cells are positive, the solution is optimal.

Hence, the optimal cost of transportation is 76.

Problem 7: (A case of unbalanced problem)

The ABC Tool company has a sales force of 25 men who work out from three regional offices. The company produces four basic product lines of hand tools. Mr. Jain, sales manager feels that 6 salesmen are needed to distribute product line 1, 10 salesmen to distribute product line 2, 4 salesmen to product line 3 and 5 salesmen to product line 4. The cost (in ₹) per day of assigning salesmen from each of the offices for selling each of the product lines are as follows.

Regional	*Product lines*			
Office	*1*	*2*	*3*	*4*
A	20	21	16	18
B	17	28	14	16
C	29	23	19	20

At the present time, 10 salesmen are allotted to office A, 9 salesmen to office B and 7 salesmen to office C. How many salesmen should be assigned from each office to selling each product line in order to minimise costs?

Solution:

In this case, we can list all the information on the matrix given

		Product lines			*Salesmen*
Offices	*1*	*2*	*3*	*4*	*available*
A	20	21	16	18	10
B	17	28	14	16	9
C	29	23	19	20	7
Salesmen required	6	10	4	5	

It can be seen that it is not a balanced problem *i.e.,* the requirement of salesman is less than the salesmen available. We, therefore, add a dummy line 5 with the requirement of 1 salesman with costs per day assigned as zero. The revised matrix; is as follows:

			Product lines			*Availability*
Offices	*1*	*2*	*3*	*4*	*5*	
A	20	21	16	18	0	10
B	17	28	14	16	0	9
C	29	23	19	20	0	7
Requirement	6	10	4	5	1	26

The initial solution as obtained by using VAM (Vogel's Approximation Method) is as shown in the table below:

	1	*2*	*3*	*4*	*5*
A	20	21 ④	16 ①	18 ⑤	0
B	17 ⑥	28	14 ③	16	0
C	29	23 ⑥	19	20	0 ①

Using MODI's method, we get the optimal solution as follows:

	1	*2*	*3*	*4*	*5*
A	20	21 ④	16 ①	18 ⑤	0
B	17 ⑥	28	14 ③	16	0
C	29	23 ⑥	19	20	0 ①

This is the optimal feasible solution and the transportation cost associated with this solution works out to

$21 \times 4 + 16 \times 1 + 18 \times 5 + 17 \times 6 + 14 \times 3 + 23 \times 6 + 0 \times 1 =$ ₹ 472

Problem 8: ***(A case of unbalanced problem)***

A company manufacturing air coolers has two plants located at Bombay and Calcutta with a weekly capacity of 200 units and 100 units respectively. The company supplies air coolers to its 4 showrooms situated at Ranchi, Delhi, Lucknow and Kanpur, which have a demand of 75, 100, 100 and 30 units respectively. The cost per unit (in ₹) is shown in the following table.

Plants	Ranchi	Delhi	Lucknow	Kanpur
Bombay	90	90	100	100
Calcutta	50	70	130	85

Plan the production programme so as to minimise the total cost of transportation?

Solution:

Bringing all the information into the standard Transportation matrix, the problem is presented in the following form:

Plants	Ranchi	Delhi	Lucknow	Kanpur	Capacity
Bombay	90	90	100	100	200
Calcutta	50	70	130	85	100
Demand	75	100	100	30	

We notice that the demand exceeds the capacity by 5 units and hence a dummy row is to be added, as it is an unbalanced problem. The revised matrix will be as follows:

Plants	Ranchi	Delhi	Lucknow	Kanpur	Capacity
Bombay	90	90	100	100	200
Calcutta	50	70	130	85	100
Dummy	0	0	0	0	5
Demand	75	100	100	30	305

By using Vogel's Approximation Method, we obtain the initial allocations as follows:

Plants	*Ranchi*		*Delhi*		*Lucknow*		*Kanpur*	
Bombay	90		90	75	100	95	100	30
Calcutta	50	75	70	25	130		85	
Dummy	0		0		0	5	0	

To test the optimality of the solution, we use MODI's method and observe that the solution arrived at by VAM is optimal. The allocation in the dummy row for Lucknow indicates that there is surplus demand. The transportation cost according to this optimal solution works out to

Total Cost = $90 \times 75 + 100 \times 95 + 100 \times 30 + 50 \times 75 + 70 \times 25 + 0 \times 5 =$ ₹ 24,750

Problem 9: (*Case of Alternate optimal solution*)

The transportation cost matrix for a given situation for supply of the commodity from sources *A*, *B, C* to the points of usage *P, Q* and *R* is given below. Work out the optimal cost solution for the problem.

	P	*Q*	*R*	*Supply*
A	4	8	8	76
B	16	24	16	82
C	8	16	24	77
Demand	72	102	41	

Solution:

The above problem can be solved by first balancing it by adding a column S to compensate for the additional demand of 20 units. Thus the problem gets converted to

	P	*Q*	*R*	*S*	*Supply*
A	4	8	8	0	76
B	16	24	16	0	82
C	8	16	24	0	77
Demand	72	102	41	20	235

By using Vogel's Approximation Method, we get the initial solution as follows :

	P		*Q*		*R*		*S*	
A	4		8	35	8	41	0	
B	16		24	62	16		0	20
C	8	72	16	5	24		0	

Now using MODI method for optimality test, we get the allocations

	P		*Q*		*R*		*S*	
A	4		8	76	8		0	
B	16		24	21	16	41	0	20
C	8	72	16	5	24		0	

Optimal cost = 8 × 76 + 24 × 21 + 16 × 41 + 0 × 20 + 8 × 72 + 16 × 5 = ₹ 2,424.

While working out the opportunity cost by U-V method, we find that while all other values are positive, the value for cell BP is zero. Hence. if we bring in cell BP into the basis, there will be no change in the transportation cost. We now work out this alternative solution and get

	P		*Q*		*R*		*S*	
A	4		8	(76)	8		0	
B	16	㉑	2		16	㊶	0	⑳
C	8	(51)	16	㉖	24		0	

Since all the values of opportunity cost are positive or zero, the solution is optimal again with the same total transportation cost.

i.e. Total cost = 16 × 21 + 8 × 76 + 16 × 41 + 0 × 20 + 8 × 51 + 16 × 26 = ₹ 2,424.

Thus this is a case of alternative optimal solution.

Problem 10: (*A case of Degeneracy*)

Solve the following problem for optimal solution:

	P	*Q*	*R*	*S*	*T*	*Supply*
A	5	8	6	6	3	8
B	4	7	7	6	5	5
C	8	4	6	6	4	9
Demand	4	4	5	4	8	

Solution:

In this case, the problem is unbalanced. Hence, firstly we have to convert it into the balanced problem by adding a dummy row as D, with costs as zeros. Thus, the matrix converts into the form.

	P	*Q*	*R*	*S*	*T*	*Supply*
A	5	8	6	6	3	8
B	4	7	7	6	5	5
C	8	4	6	6	4	9
D	0	0	0	0	0	3
Demand	4	4	5	4	8	25

Now we can obtain the initial basic solution by Vogel's Approximation Method (VAM). The soluion so obtained is as follows:

	P	*Q*	*R*	*S*	*T*
A	5	8	6 ⑤	6	3 ③
B	4 ④	7	7	6 ①	5
C	8	4 ④	6	6	4 ⑤
D	0	0	0	0 ③	0

We observe that there are only 7 allocations against $m + n - 1 = 4 + 5 - 1 = 8$. Hence, the solution is degenerate and non-feasible.

To take care of degeneracy, we allocate Î to the unoccupied cell with least cost *i.e.,* either cell *AP* or *BT*. First we assign ∈ to the cell *BT*, formulate loop and determine *Ui*s and *Vj*'s for the table. Thus, we get the matrix as follows.

	P	*Q*	*R*	*S*	*T*	*Ui's*
A	5	8	6 ⑤	6	3 ③	0
B	4 ④	7	7	6 ①	5 (∈)	2
C	8	4 ④	6	6	4 ⑤	1
D	0	0	0	0 ③	0	4
V*j*	2	3	6	4	3	

Since the largest negative value of $\Delta j = Cij - (Ui + Vj)$ occurs for cell ΔR, we bring in ΔR into the basis and *BT* leaves the basis. The revised solution so obtained is given in the matrix below.

	P	*Q*	*R*	*S*	*T*	*Ui's*
A	5	8	6 ⑤	6	3 ③	0
B	4 ④	7	7	6 ①	5 ⓔ	0
C	8	4 ④	6	6	4 ⑤	1
D	0	0	0 (∈)	0 ③	0	-6
V*j*	4	3	6	6	3	

The solution is again tested for optimality by MODI method, based on *Ui*s and *Vj*s and Δij. In this case, all the values of Δij are found to be non-negative and hence the solution optimal is given below.

	P	*Q*	*R*	*S*	*T*
A	5	8	6	6	3 ⑧
B	4 ④	7	7	6 ①	5
C	8	4 ④	6 ②	6 ③	4
D	0	0	0 ③	0	0

The minimum total cost of transportation works out to

4 × 4 + 4 × 4 + 6 × 2 + 0 × 3 + 6 × 1 + 6 × 3 + 3 × 8 = ₹ 92.

The same method can be tried out by assigning Î to the other low equal cost cell AP and solution obtained. To avoid or resolve the degeneracy during optimality test, the quantity can be allocated to one or more cells, so that we can have $m + n - 1$ allocations in the new modified solution. By various allocations to the unoccupied cells, we find the solution the same *i.e.,* the minimum cost works out to ₹ 92. Thus, this type of degeneracy has its own importance in the decision making process.

8.13 Summary

This unit is summarised by using below points:

- **Transportation Model:** A special case of linear programming to obtain optimum distribution of goods within the constraints of demand and supply of the products.
- **Balanced Situation:** When the demand and supply levels tally in the transportation matrix.
- **Degeneracy:** When the number of allocations in the solution to a transportation problem is not equal to (m+n–1) is feasible solution condition, the situation is said to be decorative.
- **Loops:** The group of allocated calls in transportation solution with the initial empty cell.
- **Stone cells:** Occupied or allocated cells in transportation matrix.
- **Water cells:** Unallocated calls in transportation matrix.

8.14 Key Words/Abbreviations

- **Loops in the Transportation Problem:** In transportation problem we get a loop when solution is not optimal
- **Testing the Optimality:** If the reduced costs of all non basic variables are negative (positive), the solution is optimal
- **Degeneracy:** If number of positive independent allocations is less than m+n-1, then Initial Basic Feasible Solution is Degenerate.
- **Constraints in Transportation Problem:** The constraints are that you must (at least) meet demand at each demand center and cannot exceed supply at each supply center.

8.15 Learning Activity

1. What are the future requirements of local transportation means as for elevators, escalators?

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2. How can we measure the level of integration in transport systems?

8.16 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. Under what condition, the initial transportation solution obtained through any of the methods becomes 'feasible'?
2. Can you proceed with the initial solution of transportation problem without balancing the matrix? If not, how do you balance it?

3. Do we follow the same procedure for solving a maximisation or minimisation case? If not, what changes are adopted for the transportation table?
4. How do you resolve a case of degeneracy in the initial or intermediate solution of the transportation matrix?

B. Multiple Choice/Objective Type Questions

1. Which of the following method can be used for solving transportation problem?

   (a) North-West Corner Rule

   (b) Least Cost Method

   (c) Vogel's Approximation Method

   (d) All of these

2. An Occupied or allocated cells in transportation matrix is called ________________.

   (a) Stone cell (b) Water cell

   (c) Loops (d) Degeneracy

3. The column, Which is introduced in the matrix to balance the Rim condition, is known as___________________.

   (a) Key column (b) Idle column

   (c) Slack column (d) Dummy column

4. In transportation problem, when demand is equal to resources is known as ___________.

   (a) Balanced transportation problem

   (b) Regular transportation problem

   (c) Resource allocation transportation problem

   (d) Simple transportation problem

5. The opportunity cost of a row in a transportation problem is obtained by _______________.

   (a) Deducting the smallest element in the row from all other elements of the row

   (b) Deducting the smallest element in the row from the next highest element of the row

   (c) Deducting the smallest element in the row from the highest element in that row

   (d) Adding the smallest element in the row to all other elements of the row.

6. MODI stands for ______________.

   (a) Model assignment information (b) Modulus of integers

   (c) Modified distribution method (d) Modal distribution method

7. In the transportation problem, the cost element of one or two cells are not given in the problem ,it means _______________.

   (a) We can allocate zeros to those cells

   (b) Allocate very high cost to those cells

   (c) Allocate very minimum cost to those cells

   (d) To assume that route connected by those cells are not available

8. In transportation cost, the opportunity cost is given by _____________________.

   (a) Implied cost + actual cost of the cell

   (b) Implied cost – actual cost of the cell

   (c) Implied cost × actual cost of the cell

   (d) Actual cost of the cell – Implied cost

9. In transportation problem number of basic cells will be equal to _______________.

   (a) $m + n - 0$ (b) $n + m - 1$

   (c) $m + n - 1$ (d) None of the above

10. For maximization in transportation problem, the objective is to maximize the total _____________.

    (a) Solution (b) cost Matrix

    (c) Profit (d) None of the above

Answers:

1. (d), 2. (a), 3. (d), 4. (a), 5. (b), 6. (c), 7. (d), 8. (b), 9. (c), 10. (c)

8.17 References

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UNIT 9 ASSIGNMENT PROBLEMS

Structure:

9.0 Learning Objectives

After studying this unit, you will be able to:

- Explain the assignment problems and their characteristics.
- Discuss the method of presentation of assignment problem.
- Elaborate the methods of solving assignment problems such as Hungarian Assignment Method (HAM) and others.
- Discuss the steps for solution of assignment problems through Hungarian Method.
- Explain the modifications to cater for various constraints for assignments.
- Describe solution of a Typical Assignment Problem, namely Travelling Salesman Problem.
- Illustrate the various solved examples.
- Analyse yourself through self assessment problems.
- Define assignment problem
- Explain special algorithm for the assignment problem

9.1 Introduction

In chapter 4, we have dealt with the Distribution or Transportation problems and we discussed that it is a linear programming problem but due to its peculiar characteristics, it can easily be solved by using transportation models such as North-West Corner Rule, least cost method or Vogel's Approximation Method. By use of these methods in a step-wise systematic approach, an optimal or near optimal solution can be arrived at. We then described various methods for testing the optimality of the solution obtained. These methods help in reaching the optimal solution in a gradual iterative manner.

We face another peculiar situation, when a particular job or machine is to be assigned to a particular worker based on his training and proficiency level or a project manager has to be deputed to a particular project based on his qualification, experience and special exposure to that special type of project.

Thus, there are many situations where the assignment of people or machine etc. may be called for. Assignment of workers to machines; clerks to various counters, salesmen to different sales areas, service crews to different aircrafts are few such examples. The assignment becomes a problem because people possess varying abilities for performing different jobs and therefore, the cost of performing the jobs by different people are different. The objective function would, therefore, be minimisation of cost or time.

9.2 Define Assignment Problem

Assignment Problem is a special type of linear programming problem where the objective is to minimise the cost or time of completing a number of jobs by a number of persons.

The assignment problem in the general form can be stated as follows:

"Given n facilities, n jobs and the effectiveness of each facility for each job, the problem is to assign each facility to one and only one job in such a way that the measure of effectiveness is optimised (Maximised or Minimised)."

Several problems of management has a structure identical with the assignment problem.

9.3 Presentation of the Assignment Problem

The problem can be illustrated by the following table—

	Jobs			
Workers	*A*	*B*	*C*	*D*
1	C_{1A}	C_{1B}	C_{1C}	C_{1D}
2	C_{2A}	C_{2B}	C_{2C}	C_{2D}
3	C_{3A}	C_{3B}	C_{3C}	C_{3D}
4	C_{4A}	C_{4B}	C_{4C}	C_{4D}

Cost or time taken by workers on various jobs is indicated in matrix cells as *Cij*.

9.4 Characteristics of the Assignment Problem

From the general description of the assignment problem and the methods described for solving such problems, we can list out following characteristics of these problems.

1. The available resources are finite in number such as availability of workers, machines, project managers, salesman, jobs etc.
2. These available resources can be assigned only on one-to-one basis i.e., job can be assigned to a particular employee only once and after this one-to-one assignment, neither the worker, nor the job so assigned is available for any further consideration.
3. The outcome or the results are expressed in terms of costs, time or profits.
4. The assignment methods aim at either cost minimisation or profit maximisation, indicating that the assignments are made with a specific purpose of either cost, time or distance reductions (minimisation) or profit or utility maximisation.
5. For one-to-one assignment, the problem has to be of the balanced type, otherwise it has to be converted into a balanced problem or into a square matrix.

9.5 Methods for Assignment Problem Solutions

1. **Transportation Model :** by making supply and demand position as 1 each.

The assignment problem usually is represented in the matrix or tabular form as given above in 5.2. This table is similar to that for transportation problem. In fact, assignment problem can be represented as transportation table in the following manner.

	Jobs				
Worker	*A*	*B*	*C*	*D*	*Supply*
1	15	20	31	27	1
2	14	16	20	15	1
3	22	24	27	19	1
4	17	15	9	12	1
Demand	1	1	1	1	

This presentation is valid because for assignment, there is only one person available for a job. Hence once worker 1 has been assigned to a job say A, no other job can be assigned to him. Hence, supply position is indicated as 1. Similarly, once a job A has been consigned to worker 1, it is not available to be assigned to any other worker, hence demand position 1.

2. **Simplex Method :** by making objective function expressing total cost or profit and then by deciding whether to minimise or maximise it.

Such as, Minimise $Z = \sum \sum Cij\ Xij$, where *Cij* is the cost associated for an assignment from *i* to *j*.

Subject to, $\sum_{i=1}^{n} Xij = 1$ (supply position)

$\sum_{i=1}^{m} Xij = 1$ (demand position)

and $Xij = 0$, or 1 for all values of *i*'s and *j*'s because there can be only one assignment in each supply constraint or one in each demand constraint.

Where in, Cij = Cost of performing the jobs *j* by worker *i*.

Xij = Number of workers *i* performing jobs *j*.

Above problem, therefore, can be formulated as an LP problem taking the above given matrix as cost matrix.

$$\begin{aligned}\text{Min. Ƶ} = {} & C_{1A} . X_{IA} + C_{1B} . X_{1B} + C_{1C} . X_{1C} + C_{ID} . X_{ID} + C_{2A} . X_{2A} + C_{2B}. X_{2B} + C_{2C} . X_{2C} + \\ & C_{2D}.X_{2D} + C_{3A}. X_{3A} + C_{3B}. X_{3B} + C_{3C}. X_{3C\cdot} + C_{3D}. X_{3D} + C_{4A}. X_{4A} + C_{4B}. X_{4B\cdot} + \\ & C_{4C}. X_{4C\cdot} + C_{4D}. X_{4D}\end{aligned}$$

Subject to, $X_{1A} + X_{1B} + X_{1C} + X_{1D} = 1$ etc.

3. **Branch and Bound Method :** Due to its integer characteristics, the method of integer programming can be used as described in Chapter 20.

4. **Complete Enumeration Method :** In all, there can be 4! = 24 assignments for a 4 × 4 matrix situation. We work out cost for all assignments.

Cost for all assignments can be calculated for following combinations

1. 1A, 2B, 1C, 1D
2. 1A, 2B, 1C, 1D
3. 1A, 3D, 1C, 1D
4. 1A, 4B, 1C, 1D
5. 2A, 1B, 1C, 1D
6. 2A, 2B, 1C, 1D

7. 2A, 3B, 1C, 1D
8. 2A, 3B, 2C, 1D
9. 3A, 3B, 3C, 2D
10. 3A, 3B, 3C, 3D etc.

This is also a tedious job as enumeration can be very large.

5. **Hungarian Assignment Method (HAM) :** When the objective function is that of minimisation type, we follow the steps given below, after ensuring square matrix. If it is not a square matrix, a row/column is to be added with all zero elements. It is called a dummy row/column. The method of solution is given in the next paragraph.

9.6 Hungarian Assignment Method (HAM)

Step 1. Locate the smallest cost element in each row of the cost matrix. Then subtract this smallest element from each element in that row. As a result, there shall be atleast one zero in each row of the new matrix.

Step 2. Now consider each column of the reduced cost matrix from step 1 and locate smallest element in it. Subtract the smallest value from each element of the column. There would, again, be at least one zero in each column of the Second Reduced cost matrix.

Step 3. Draw *minimum* number of horizontal and vertical lines to cover *all* zero elements. If the number of lines drawn is equal to the number of rows/columns (*n*), the solution is *optimal.* Proceed to step 6. If number of lines is less than the number of rows/columns, go to step 4.

Step 4. Select smallest uncovered cost element of the modified matrix from step 3. Subtract this element from all uncovered elements and add it to each value located at the interactions of any two lines.

Step 5. Repeat step 3 and 4 till optimal solution is obtained i.e., number of lines drawn equals number of column/rows.

Step 6. Make feasible job assignments on zero elements.

These steps can now be seen in the flow chart drawn as Fig. 9.1.

9.7 Flow Chart of Assignment Problem - Hungarian Assignment Method

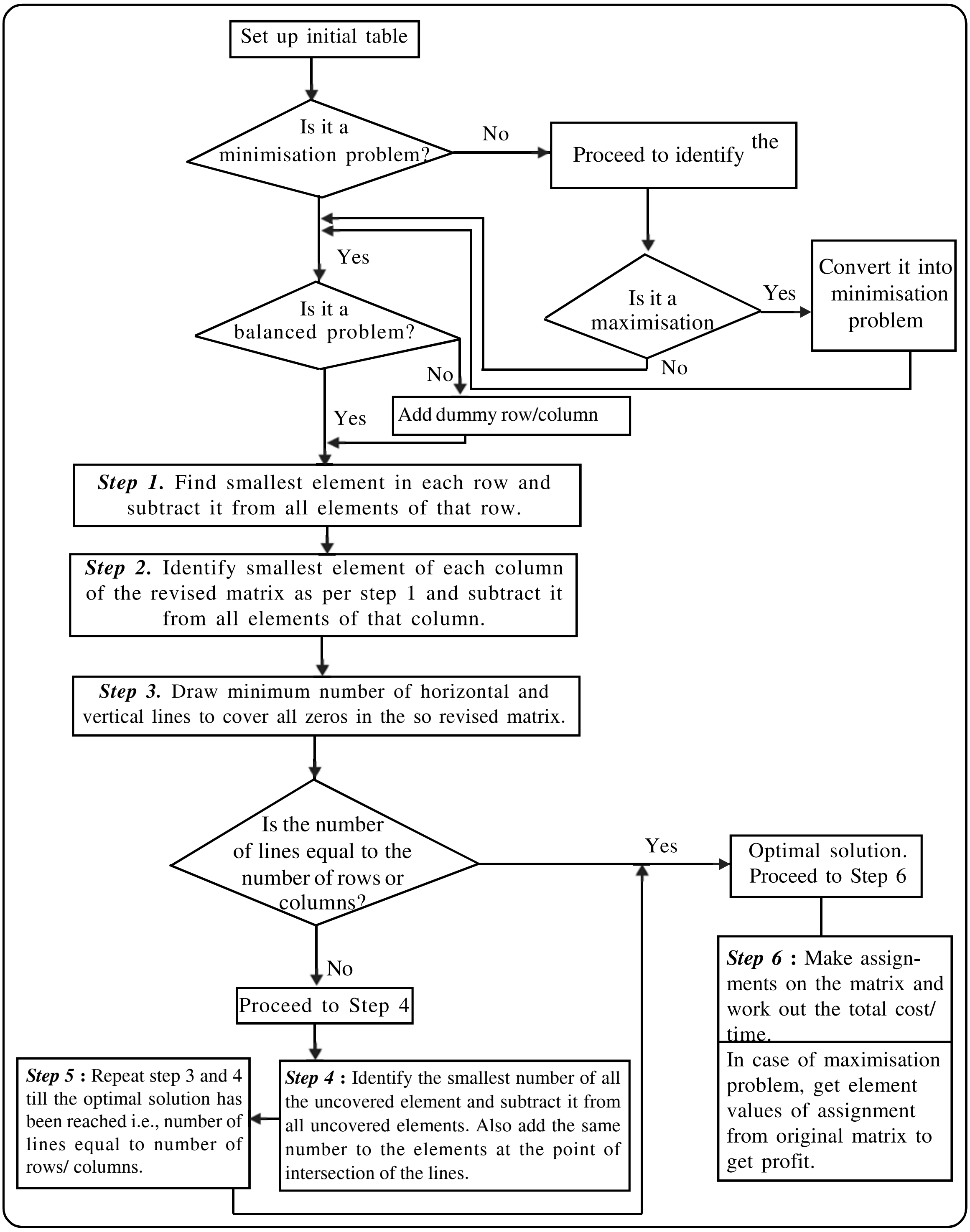
Set up initial table
Is it a minimisation problem?
No
Proceed to identify the
Is it a maximisation
Yes
Convert it into minimisation problem
No
Yes
Is it a balanced problem?
No
Add dummy row/column
Yes
Step 1. Find smallest element in each row and subtract it from all elements of that row.
Step 2. Identify smallest element of each column of the revised matrix as per step 1 and subtract it from all elements of that column.
Step 3. Draw minimum number of horizontal and vertical lines to cover all zeros in the so revised matrix.
Is the number of lines equal to the number of rows or columns?
Yes
Optimal solution. Proceed to Step 6
Step 6 : Make assignments on the matrix and work out the total cost/ time.
In case of maximisation problem, get element values of assignment from original matrix to get profit.
No
Proceed to Step 4
Step 4 : Identify the smallest number of all the uncovered element and subtract it from all uncovered elements. Also add the same number to the elements at the point of intersection of the lines.
Step 5 : Repeat step 3 and 4 till the optimal solution has been reached i.e., number of lines equal to number of rows/ columns.

Fig. 9.1

9.8 Constrained Assignment Problems

Following types of constraints arise, while dealing with assignment problems.

Constraint of Unbalancing and Prohibitive Assignment

If rows and column of assignments problem are not equal in number, it is called an unbalanced problem. In such as case, a dummy row/column can be added with all cost elements as zero. There can be yet another constraint in the problem wherein a particular assignment is not desirable for whatever reason. In that case, the cost of assignment in that cell can be made prohibitive, thus writing it as M, i.e., excessively high value and unassignable. Problem is then solved in the usual manner.

This can be illustrated by taking up a problem 9.3.

Constraint of Maximisation Situation

Standard Hungarian Method (HAM) deals with minimisation situations. When the data given is that of maximisation, the methodology undergoes slight modification.

1. We convert maximisation problem into minimisation situation by deducting all the elements of the matrix from the highest element of the original pay-off matrix. The resultant matrix can then be used for solution by Hungarian Assignment Method. In the final stage, the values from the original matrix are to be used for finding out the optimal value of the decision parameters, such as assigned value of profit.
2. Instead of using maximum value of the matrix for subtraction from matrix elements, we can use minimum value of the matrix. By subtracting it from all elements, we can obtain matrix for minimisation case to be treated with Hungarian Assignment Method. For optimal solution value, original matrix values are to be utilised.
3. We can also convert the maximisation problem into minimisation by multiplying all its elements by (-1) and then using standard HAM.
4. In either cases M (prohibitive element) is not considered in these applications.

Constraint of Multiple Optimal Solution

While assigning at zero elements, there may be more than one feasible way and still over-all pay-off effect may be the same. Then management has the liberty to decide on merit or personal experience.

Travelling Salesman Problem

A salesman is required to travel within his allotted territory wherein he has certain number of cities to visit and he has to plan these visits in the most cost effective manner. He has a base city (say, where he resides) and he wishes to visit each city only once, before returning to the base. Since the distance of journey between various pair of cities is known, it is only optimising this distance travelled that a salesman will aim for. The problem could also be that of minimising cost or time of travel.

This problem can be called a case of zero-one programming or Integer linear programming, because salesman either travels to a city or does not travel. It can be treated as a transportation problem with a square matrix and $Xij = 1$ or 0. But there are only (n - 1) possible ways for the journey and it is not easy to solve by transportation method. We use Assignment Method for such problem, with a restriction, that $Cij = 0$ when $i = j$, *i.e.*, the salesman cannot travel from a city to the same city.

Various problems are solved for all these constraints in turn.

9.9 Explain Special Algorithm for the Assignment Problem

The assignment problem is a special case of the transportation problem, which in turn is a special case of the min-cost flow problem, so it can be solved using algorithms that solve the more general cases. Also, our problem is a special case of binary integer linear programming problem (which is NP-hard).

Core of the algorithm (assuming square matrix):

1. For each row of the matrix, find the smallest element and subtract it from every element in its row.
2. Do the same (as step 1) for all columns.

3. Cover all zeros in the matrix using minimum number of horizontal and vertical lines.
4. *Test for Optimality:* If the minimum number of covering lines is n, an optimal assignment is possible and we are finished. Else if lines are lesser than n, we haven't found the optimal assignment, and must proceed to step 5.
5. Determine the smallest entry not covered by any line. Subtract this entry from each uncovered row, and then add it to each covered column. Return to step 3.

9.10 Solved Examples

Problem 1 :

Assign workers 1, 2, 3, 4 to jobs A, B, C, D . Time taken by workers for different jobs are given in the matrix:

	Jobs			
Workers	*A*	*B*	*C*	*D*
1	45	40	51	67
2	55	40	61	53
3	49	52	48	64
4	41	45	60	55

Solution :

Step 1. Row minima have been identified. These values are to be subtracted from all values of the respective rows.

		Jobs			*Row minima*
Workers	*A*	*B*	*C*	*D*	
1	45	40	51	67	40
2	55	40	61	53	40
3	49	52	48	64	48
4	41	45	60	55	41

Reduced time matrix is obtained thus—

Workers *Jobs*	*A*	*B*	*C*	*D*
1	5	0	11	27
2	15	0	21	13
3	1	4	0	16
4	0	4	19	14
Column minima	0	0	0	13

Revised time matrix I

Step 2. Now we obtain minimum values of each column from the above reduced time matrix I and subtract these from each respective column elements to achieve revised matrix II.

Workers *Jobs*	*A*	*B*	*C*	*D*
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	1	19	1

Revised time matrix II

Step 3. Now drawing minimum number of horizontal/vertical lines to cover all zero elements, we get the matrix III below.

Workers *Jobs*	*A*	*B*	*C*	*D*
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	4	19	1

Since the number of lines drawn is 4 = *n*, the solution is optimal. As can be seen, the line on row 2 and column B cover 2 zeros each, whereas to cover remaining 2 zeros, two lines on row 3 and 4 have to be drawn. Minimum 4 lines cover all zeros. Hence, step 4 and 5 are not required.

Proceed to step 6.

Step 6. Making assignments on zero elements, we obtain

Workers / *Jobs*	*A*	*B*	*C*	*D*
1	5	0	11	14
2	15	0	21	0
3	1	4	0	3
4	0	4	19	1

Final time matrix and assignments

Row 1 has only one unique zero at element 1B. Hence job B has been assigned to worker 1. Similarly row 4 has only one zero at 4A, hence job A assigned to worker 4. Job B having been assigned, zero at 2B has become ineffective. Now row 2 is left with only one zero at location 2D. Hence, job D has been assigned to worker 2. Similarly job C can be assigned to worker 3. Hence, jobs have been assigned as indicated by squares drawn on zero elements 4A, 1B, 3C, 2D.

Total time = 41 + 40 + 48 + 53 = 182 min (optimal)

This can be compared by performing complete enumeration method. The minimum time obtained will be 182 only.

Problem 2 :

Five men are available to do five different jobs. From past records, the time (in hours) that each man takes to do a job is known and is given in the following matrix.

			Jobs		
Men	*I*	*II*	*III*	*IV*	*V*
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

Find the assignment of men to jobs that will minimise the total time taken.

Solution :

Check 1. whether minimisation problem—Yes

2. whether square matrix—Yes

Then proceed to apply Hungarian Assignment Method to solve it.

Step 1. Identify the smallest element of each row and subtract it from all elements of that row. Revised matrix is as given below. (Row minima for row A, B, C, D, E = 1)

			Jobs			
Men	*I*	*II*	*III*	*IV*	*V*	
A	1	8	1	6	0	
B	5	7	6	5	0	
C	3	5	4	2	0	Revised
D	3	1	6	2	0	matrix I
E	4	2	8	4	0	

Step 2. Identify the smallest element of each column and subtract it from all the elements of the column.

Column minimas are Column I ——→ 1

Column II ——→ 1

Column III ——→ 1

Column IV ——→ 2

Column V ——→ 0

Hence, the reduced matrix works out to:

			Jobs			
Men	*I*	*II*	*III*	*IV*	*V*	
A	0	7	0	4	0	
B	4	6	5	3	0	
C	2	4	3	0	0	Revised
D	2	0	5	0	0	matrix II
E	3	1	7	2	0	

Step 3. Draw minimum number of horizontal and vertical lines to cover all zeros.

			Jobs		
Men	*I*	*II*	*III*	*IV*	*V*
A	0	7	0	4	0
B	4	6	5	3	0
C	2	4	3	0	0
D	2	0	5	0	0
E	3	1	7	2	0

Since we have been able to over all zeros with 4 lines, which is less than the number of rows/ columns *i.e.* 5, the solution is not optimal. Hence, proceed to step 4.

Step 4. We identify the smallest uncovered element as 2. Subtracting 2 from all uncovered elements and adding it to the elements at the point of intersection of lines, we get the revised matrix as

			Jobs			
Men	*I*	*II*	*III*	*IV*	*V*	
A	0	9	0	6	2	
B	2	6	3	3	0	
C	0	4	1	0	0	Revised
D	0	0	3	0	0	matrix III
E	1	1	5	2	0	

Step 5. Repeating step 3, lines drawn are as follows

			Jobs			
Men	*I*	*II*	*III*	*IV*	*V*	
A	0	9	0	6	2	
B	2	6	3	3	0	
C	0	4	1	0	0	Revised
D	0	0	3	0	0	matrix IV
E	1	1	5	4	0	

Since number of lines drawn are again 4 ≠ number of rows/columns, repeat step 4. Minimum uncovered element is 1. Subtracting 1 from all uncovered elements and adding it to the elements at point of intersection of lines, revised matrix will be

			Jobs			
Men	*I*	*II*	*III*	*IV*	*V*	
A	0	9	0	6	3	
B	1	5	2	2	0	Revised
C	0	4	1	0	1	matrix V
D	0	0	3	0	1	
E	0	0	4	3	0	

Repeating step 3, number of lines drawn are 5 = no. of rows/columns. Hence, we arrive at the optimal solution. Proceed to step 6.

Step 6. Making assignments on zero elements

			Jobs			
Men	*I*	*II*	*III*	*IV*	*V*	
A	$\cancel{0}$	9	$\boxed{0}$	6	3	
B	1	5	2	2	$\boxed{0}$	Final matrix
C	$\cancel{0}$	4	1	$\boxed{0}$	1	for
D	$\cancel{0}$	$\boxed{0}$	3	$\cancel{0}$	1	assignments
E	$\boxed{0}$	$\cancel{0}$	4	3	$\cancel{0}$	

Hence, minimum total time = 2 + 1 + 3 + 2 + 5 = 13 units (hours).

This is a case of multiple assignments, where assignment A III and B V are unique. But even after cancelling zeros at A I and E V, we are left with 3 zeros in column I, 2 in column II and 2 in column IV. Thus if we assign C IV, the zeros at C I and D IV get eliminated. Even now, we have 2 zeros in column I and II. These can be assigned alternatively as E I, D II or D I, E II, etc.

Problem 3 : (A case of unbalanced problem and prohibitive assignment.)

Let us consider a problem of assigning the jobs to four (4) persons, whereas the jobs available are five (5). The costs of working by various persons for particular jobs are indicated in the matrix form (it can also be time taken by a person for performing a particular job). Wherever the job cannot be assigned to any particular person, the value is indicated as cross (×) or Big M (M).

Matrix of the problem is as follows :

			Jobs		
Persons	J_1	J_2	J_3	J_4	J_5
P_1	26	18	x	20	21
P_2	31	24	21	12	17
P_3	20	17	20	x	16
P_4	20	28	20	16	27

Solution :

We substitute X by M (Prohibitive) and add a dummy row P_5 for balancing the problem. Matrix so obtained can now be treated using Hungarian Assignment Method.

The matrix, thus, takes the following values :

	Jobs						
Persons	J_1	J_2	J_3	J_4	J_5	*Row minima*	
P_1	27	18	M	20	21	18	
P_2	31	24	21	12	17	12	
P_3	20	17	20	M	16	16	
P_4	20	28	20	16	27	16	(dummy
P_5	0	0	0	0	0	0	row)

Step 1. Subtracting Row minima from each element of that row, we get revised cost matrix.

	Jobs				
Persons	J_1	J_2	J_3	J_4	J_5
P_1	9	0	M	2	3
P_2	19	12	9	0	5
P_3	4	1	4	M	0
P_4	4	12	4	0	11
P_5	0	0	0	0	0

Step 2. Minima of each column is zero and subtracting this from all that elements of the columns, will leave the matrix as it is.

Step 3. Drawing Minimum number of Horizontal and Vertical lines to cover all zeros in the matrix, we get

	Jobs				
Persons	J_1	J_2	J_3	J_4	J_5
P_1	9	0	M	2	3
P_2	19	12	9	0	5
P_3	4	1	4	M	0
P_4	4	12	4	0	11
P_5	0	0	0	0	0

We get 4 lines, whereas row/columns are 5. Hence solution is not optimal. Proceed to step 4.

[*Note* : First horizontal line drawn is row P_5. Then we find 2 zeros in column J_4 and line drawn. Now having been left with zeros at elements P_1 J_2 and P_3 J_5, we cover these zero by two vertical lines through J_2 and J_5. Hence we have covered all the zeros with minimum 4 lines drawn.]

Step 4. Minimum uncovered element is 4. (Elements uncovered by 4 lines drawn in Step 3 are 9, 19, 4, 4, M, 9, 4, 4 out of which minimum element is 4). Hence subtract 4 from all uncovered elements and add the same number 4 to the point of intersection of lines (point of intersection of lines are $P_5\, J_2$, $P_5\, J_4$ and $P_5\, J_5$).

	Jobs					
Persons	J_1	J_2	J_3	J_4	J_5	
P_1	5	0	M	2	3	
P_2	15	12	5	0	5	
P_3	0	1	0	M	0	Revised cost
P_4	0	12	0	0	11	matrix
P_5	0	4	0	4	4	

Step 5. (repeat step 3)

Drawing minimum number of horizontal and vertical lines to cover all zeros of the matrix, we get

	Jobs				
Persons	J_1	J_2	J_3	J_4	J_5
P_1	5	0	M	2	3
P_2	15	12	5	0	5
P_3	0	1	0	M	0
P_4	0	12	0	0	11
P_5	0	4	0	4	4

Since number of lines is equal to the order of the matrix i.e., equal to the number of rows/ columns, the solution is optimal and hence assignments are now made on zero elements.

Step 6.

	Jobs				
Persons	J_1	J_2	J_3	J_4	J_5
P_1	5	0	M	2	3
P_2	15	12	5	0	5
P_3	0	1	0	M	0
P_4	0	12	0	0	11
P_5	0	4	0	4	4

Assignments made are

Person P_1 to Job J_2

P_2 to Job J_4

P_3 to Job J_5

P_4 to Job J_1

Since P_5 is non-existent, this assignment is superfluous and Job 3 remains unassigned.

Optimal assignment Cost = 18 + 12 + 16 + 20 = 66 units.

This is a case of multiple solution when P_4 can be assigned J_2 and P_5 to J_1 (dummy).

Problem 4 : (A case of maximisation constraints)

A marketing manager has five salesmen and five sales districts. Considering the capabilities of the salesmen and nature of districts, the marketing manager estimates that sales per month (in hundred rupees) for each salesman in each district would be as follows—

	Districts				
Salesmen	*A*	*B*	*C*	*D*	*E*
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to districts that will result in maximum sales.

Solution :

Check 1. Is it the balanced problem—Yes.

2. Is it the minimisation problem—No.

Then we convert it to standard minimisation problem by subtracting all the elements from the highest value element of the matrix i.e., 41. The minimisation matrix is thus as given below:

9	3	1	13	1
1	17	13	20	5
0	14	8	11	4
19	3	0	5	5
12	8	1	6	2

Revised matrix
(minimisation type)

Using HAM for solving the revised matrix, we proceed as follows.

Step 1. Row minima being subtracted from all elements of the row, the revised matrix will become

8	2	0	12	0
0	16	12	19	4
0	14	8	11	4
19	3	0	5	5
11	7	0	5	1

Matrix I

Step 2. Selecting column minima and subtracting it from all the elements of that column in the matrix, it is converted to,

8	0	0	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	0	5
11	5	0	0	1

Matrix II

Step 3. Drawing minimum number of horizontal and vertical lines across all zeros

8	0	0	7	0
0	14	12	14	4
0	12	8	6	4
19	1	0	0	5
11	5	0	0	1

Since there are only 4 lines (less than the number of rows/columns i.e., 5), the solution is not optimal. Proceed to step 4.

Step 4. Identifying minimum uncovered element (4), subtracting it from all the uncovered elements and adding the same at the point of intersection of the lines, the new matrix works out to be

12	0	0	7	0
0	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

Step 5. Repeating step 3, we get

12	0	0	7	0
0	10	8	10	0
0	8	4	2	0
23	1	0	0	5
15	5	0	0	1

Here, number of lines = number of rows/columns. Hence, solution is optimal. Proceed to step 6.

Step 6. Assigning on zero elements

	A	*B*	*C*	*D*	*E*
1	12	$\boxed{0}$	$\xcancel{0}$	7	$\xcancel{0}$
2	$\xcancel{0}$	10	8	10	$\boxed{0}$
3	$\boxed{0}$	8	4	2	$\xcancel{0}$
4	23	1	$\boxed{0}$	$\xcancel{0}$	5
5	15	5	$\xcancel{0}$	$\boxed{0}$	1

or,

	A	*B*	*C*	*D*	*E*
1	12	$\boxed{0}$	$\xcancel{0}$	7	$\xcancel{0}$
2	$\boxed{0}$	10	8	10	$\xcancel{0}$
3	$\xcancel{0}$	8	4	2	$\boxed{0}$
4	23	1	$\boxed{0}$	$\xcancel{0}$	5
5	15	5	$\xcancel{0}$	$\boxed{0}$	1

The assignment of salesmen to districts are multiple solutions.

1 → B		1 → B		1 → B
2 → E	or,	2 → A	or,	2 → A
3 → A		3 → E		3 → E
4 → C		4 → C		4 → D
5 → D		5 → D		5 → C

Picking the values of sales from the original matrix, Maximum sales = 38 + 36 + 41 + 41 + 35 = 191 in hundred rupees for all the assignments. Thus, there are multiple solution for the problem, the sales remaining as ₹ 19,100.

Problem 5 : (Prohibitive constraint and unbalanced case)

In the modification of a plant layout of a factory, four new machines M_1, M_2, M_3, M_4 are to be installed in a machine shop. There are five vacant places *A, B, C, D* and *E* available. Because of the limited space, machine M_2 cannot be placed at *C* and M_3 cannot be placed at *A*. The cost of locating machine *i* to place *j* (in ₹) is shown below

	Places				
Machines	*A*	*B*	*C*	*D*	*E*
M_1	9	11	15	10	11
M_2	12	9	—	10	9
M_3	—	11	14	11	7
M_4	14	8	12	7	8

Find the optimal assignment schedule.

Solution :

(*i*) The problem is of minimisation type.

(*ii*) The problem is unbalanced, hence a dummy row M_5 has been added with all cost elements as zeros.

(*iii*) Restrictions as machine M_2 and M_3 for space at *C* and *A* respectively are taken care of by making elements M_2C and M_3A as *M* (prohibitive). Now the Hungarian Method can be applied to solve this problem.

Step 1 and 2. Applying row and column minima operations on the original matrix, the reduced matrix is obtained as follows :

0	2	6	1	2
0	M	1	0	
M	4	7	4	0
7	1	5	0	1
0	0	0	0	0

Step 3. Drawing minimum number of horizontal/vertical lines to cover all zeros, the situation is represented as follows :

0	2	6	1	2
3	0	M	1	0
M	4	7	4	0
7	1	5	0	1
0	0	0	0	0

The number of lines are equal to number of rows/columns. Hence, this is optimal solution. Proceed to step 6.

Step 6. Making assignments on zero elements.

	Places				
Machines	*A*	*B*	*C*	*D*	*E*
M_1	0	2	6	1	2
M_2	3	0	M	1	0
M_3	M	4	7	4	0
M_4	7	1	5	0	1
M_5	0	0	0	0	0

The assignment made and resultant costs are as under:

Machine	*Place*	*Cost (₹)*	
M_1	*A*	9	
M_2	*B*	9	
M_3	*E*	7	
M_4	*D*	7	
M_5	*C*	0	(dummy)
		Total cost = ₹ 32	

Problem 6 : (Prohibitive Constraint)

Five workers are available to work with the machines and the respective cost (in ₹) associated with each worker-machine assignment is given below. A sixth machine is available to replace one of the existing machines and the associated costs are also given as follows :

Machine

Worker	M_1	M_2	M_3	M_4	M_5	M_6
W_1	12	3	6	—	5	0
W_2	4	11	—	5	—	3
W_3	8	2	10	9	7	5
W_4	—	7	8	6	12	10
W_5	5	8	9	4	6	—

(*a*) Determine, whether the new machine can be accepted.

(*b*) Determine also optimal assignments and associated saving in cost.

Solution :

(*a*) To convert the matrix into a balanced one, a row is added as dummy with all its elements as zero and restricted assignments represented by (–) are replaced by M (prohibitive). Now Hungarian Method can be used for this minimisation problem.

Step 1 and 2. Operations row minima and column minima are carried out and reduced matrix is given below:

0	0	3	M	2	6
1	8	M	2	M	0
6	0	8	7	5	3
M	1	2	0	6	4
1	4	5	0	2	M
0	0	0	0	0	0

Step 3. Drawing horizontal and vertical lines to cover all zeros, the situation is as given above. Since the number of lines drawn are not equal to the number of rows/columns i.e., 4 ^{1} 6, proceed to step 4.

Step 4. Minimum uncovered element is 1. This is subtracted from all uncovered elements and added to the point of intersection of the lines. The new matrix obtained is as follows :

8	0	2	M	1	5
1	9	M	3	M	0
5	0	7	7	4	2
M	1	1	0	5	3
0	4	4	0	1	M
0	1	0	1	0	0

Step 5. Repeating step 3 to reach the optimal solution, the matrix is obtained as given below:

$$\begin{array}{|cccccc|}
7 & 0 & 1 & M & 0 & 4 \\
1 & 10 & M & 3 & M & 0 \\
4 & 0 & 6 & 6 & 3 & 1 \\
M & 2 & 1 & 0 & 5 & 3 \\
0 & 5 & 4 & 0 & 1 & M \\
0 & 2 & 0 & 1 & 0 & 0
\end{array}$$

Step 6. The number of lines being 6 = number of rows/columns, this is optimal solution and assignments are made as follows:

Worker	*Machine*	*Cost (₹)*	
W_1	M_5	5	
W_2	M_6	3	
W_3	M_2	2	
W_4	M_4	6	
W_5	M_1	5	
W_6	M_3	0	(dummy worker)
		Total cost ₹ 21	

(*b*) ***Alternate solution :*** If sixth machine is not accepted, then the problem becomes balanced for 5 workers and 5 machines, and we have no assignment for sixth machine. The minimisation problem now can be solved by Hungarian Assignment Method in the usual manner. The optimal solution is obtained as the following matrix:

$$\begin{array}{|ccccc|}
9 & 0 & 1 & M & 0 \\
0 & 7 & M & 1 & M \\
6 & 0 & 6 & 7 & 3 \\
M & 1 & 0 & 0 & 4 \\
1 & 4 & 3 & 0 & 0
\end{array}$$

The assignments obtained are as follows :

Worker	*Machine*	*Cost (₹)*
W_1	M_5	5
W_2	M_1	4
W_3	M_2	2
W_4	M_3	8
W_5	M_4	4
		Total cost = ₹ 23

It is seen that by not accepting the sixth machine, the optimum operating cost has increased from ₹ 21 to ₹ 23. Hence, machine six must be accepted to reduce the cost.

Thus saving = ₹ 23 - 21 = ₹ 2.

Problem 7 : (Maximisation problem)

An airline company has drawn up a new flight schedule involving 5 flights. To assist in allocating five pilots to the flights, it has asked them to state their preference score by giving each flight a number out of 10. The higher the number, the greater is the preference. Certain of these flights are unsuitable to some pilots owing to domestic reasons. These have been marked with X.

	Flight Number				
Pilot	*1*	*2*	*3*	*4*	*5*
A	8	2	X	5	4
B	10	9	2	8	4
C	5	4	9	6	X
D	3	6	2	8	7
E	5	6	10	4	3

What should be the allocation of the pilots to flights in order to meet as many preferences as possible.

Solution :

This problem is that of maximisation type. Hence, we first convert it to the standard minimisation type by subtracting all the matrix elements from the highest score elements i.e., 10. Replacing X by M, the reduced score matrix will be as follows :

	1	*2*	*3*	*4*	*5*
A	2	8	M	5	6
B	0	1	8	2	6
C	5	6	1	4	M
D	7	4	8	2	3
E	5	4	0	6	7

Now we apply Hungarian Assignment Method for solving it. Going through various steps we get the matrix as follows:

	1	*2*	*3*	*4*	*5*
A	0	5	M	3	3
B	0	0	8	2	5
C	4	4	0	3	M
D	5	1	6	0	0
E	5	3	0	6	6

Here, Number of lines ≠ number of rows/columns.

We obtain the following matrix by operating final step 5.

	1	*2*	*3*	*4*	*5*
A	0	5	M	3	3
B	$\boxed{0}$	0	11	2	5
C	$\cancel{1}$	$\boxed{1}$	$\cancel{0}$	$\boxed{0}$	M
D	5	1	9	$\cancel{0}$	$\boxed{0}$
E	2	$\cancel{0}$	$\boxed{0}$	3	3

Since here number of lines = number of rows/columns, the solution is optimal. Hence, the assignments are marked in step 6 around zeros indicated in the square. The optimal assignment would, therefore, be

Pilot	*Flight*	*Pref. Score*
A	1	8
B	2	9
C	4	6
D	5	7
E	3	10
		Total = 40

Problem 8 : (Airline Crew Assignment)

An airline, that operates seven days a week, has a time table shown below. Crews must have a minimum layover of 6 hours between flights. Obtain the pairing of flights that minimises layover time away from home. For any given pairing, the crew will be based at the city that results in the smaller layover.

Flight	*Delhi*	*Calcutta*	*Flight*	*Calcutta*	*Delhi*
	Depart	*Arrive*		*Depart*	*Arrive*
1	7.00 AM	9.00 AM	101	9.00 AM	11.00 AM
2	9.00 AM	11.00 AM	102	10.00 AM	12.00 Noon
3	1.30 PM	3.30 PM	103	3.30 PM	5.30 PM
4	7.30 PM	9.30 PM	104	8.00 PM	10.00 PM

For each pair also, mention the town where the crew should be based.

Solution :

Let us first construct the table for the possible layovers between flights, when crews are based at Delhi. The time difference between flight 1 and 101 is 24 hrs. i.e., 1,440 minutes, whereas minimum layover required is 6 hours or 360 minutes.

When crew is based at Delhi, the layover table will be as follows :

Flights	*101*	*102*	*103*	*104*
1	1440	1500	390	660
2	1320	1380	1710	540
3	1050	1110	1440	1710
4	690	750	1080	1350

Similarly, we now construct layover table for crews based at Calcutta.

Flights	*101*	*102*	*103*	*104*
1	1200	1140	810	540
2	1320	1260	930	660
3	1590	1530	1200	930
4	510	450	1560	1290

As per the given constraint, minimum layover time is now given in the table below.

Flights	*101*	*102*	*103*	*104*
1	1200	1140	390	540
2	1320	1260	930	540
3	1050	1110	1200	930
4	510	450	1080	1290

The figures circled indicate layover for crew based at Calcutta, whereas not-circled figures are for Delhi based crew.

Step 1. Subtracting the smallest element of each row from every element of the corresponding row, we get the following:

Flights	*101*	*102*	*103*	*104*
1	810	750	0	150
2	780	720	390	0
3	120	180	270	0
4	60	0	630	840

Step 2. Subtracting column minima from all the elements of the columns.

	101	*102*	*103*	*104*
1	750	750	0	150
2	720	720	390	0
3	60	180	270	0
4	0	0	630	840

Step 3. Drawing minimum number of horizontal and vertical lines to cover all zeros, we get only 3 lines as given above. Since this number is not equal to the number of rows/columns, the solution is not optimal. Proceed to step 4.

Step 4. Identify the smallest uncovered element and subtracting it from all uncovered elements, with addition to the elements at points of intersection, the matrix will be revised as follows (Min. element = 60).

	101	*102*	*103*	*104*
1	690	690	0	150
2	660	660	390	0
3	0	120	270	0
4	0	0	690	900

Step 5. *(repeat step 3).* Drawing horizontal/vertical lines to cover all zeros, we get

	101	*102*	*103*	*104*
1	690	690	0	150
2	660	660	390	0
3	0	120	270	0
4	0	0	690	900

Number of lines are now equal to number of rows/columns (i.e., 4). Hence, the solution is optimal. Proceed to step 6.

Step 6. Making assignments on zero elements.

	101	*102*	*103*	*104*
1	690	690	$\boxed{0}$	150
2	660	660	390	$\boxed{0}$
3	⓪	120	270	0
4	0	$\boxed{0}$	690	900

Assignments are marked by

Hence, optimum assignment of crews is as follows :

Flights	*Crew Base*
3—101	Delhi
4—102	Calcutta
1—103	Delhi
2—104	Delhi

Total layover time = 390 + 540 + 1050 + 450 = 2430 minutes or 40 hrs. 30 minutes.

9.11 Self-Assessment Problems

1. Find the optimal assignment for the following cost matrix.

	Areas			
Salesmen	A_1	A_2	A_3	A_4
S_1	11	17	8	16
S_2	9	7	12	10
S_3	13	16	15	12
S_4	14	10	12	11

2. A sales manager has to assign salesmen to four territories. He has four candidates of varying experience and capabilities and assesses the possible profit for each salesman in each territory as given below. Find the assignment which maximises profit.

	Territories			
Salesmen	*A*	*B*	*C*	*D*
1	35	27	28	37
2	28	34	29	40
3	35	24	32	33
4	24	32	25	82

3. There are 5 work centres in a production shop and there are four (4) workmen, who can be deployed on these work centres. The study carried out shows that time taken by various workmen on these centres are much at variance and therefore a judicious selection and allocation of workmen is needed to get optimum production. The table showing time taken by workmen is given as follows :

	Workmen			
Work Centres	W_1	W_2	W_3	W_4
C_1	10	9	8	12
C_2	3	4	5	2
C_3	25	20	14	16
C_4	7	9	10	9
C_5	18	14	16	25

Work out the optimum allocation for various work centres.

4. The cost of performance of Jobs by various persons is given in the matrix below.

	Jobs				
Persons	J_1	J_2	J_3	J_4	J_5
P_1	10	3	3	2	8
P_2	9	7	8	2	7
P_3	7	5	6	2	4
P_4	3	5	8	2	4

Find the most cost-effective assignment of jobs to various persons.

5. A Salesman has to visit five cities A, B, C, D and E. The distances (in hundred kms,) between the five cities are as follows:

	To City				
From City	*A*	*B*	*C*	*D*	*E*
A	—	2	1	6	5
B	2	—	2	5	4
C	1	2	—	7	5
D	6	5	7	—	2
E	5	4	2	5	—

If the salesman starts from city A and has to come back to city A, which route should he select so that total distance travelled by him is minimised?

6. Suggest optimum assignment of 4 workers A, B, C and D to 4 jobs I, II, III and IV. The time taken by different workers in completing the different jobs is given below :

	Jobs			
Workers	*I*	*II*	*III*	*IV*
A	8	10	12	16
B	11	11	15	8
C	9	6	5	14
D	15	14	9	7

Also indicate the total time taken in completing the jobs.

7. A company has to assign four workers A, B, C and D to four job W, X, Y and Z. The cost matrix is given below:

	Jobs (Cost in ₹)			
Workers	*W*	*X*	*Y*	*Z*
A	1000	1200	400	900
B	600	500	300	800
C	200	300	400	500
D	600	700	300	1000

Suggest an optimal assignment schedule and the total cost pertaining thereto.

8. A large oil company operating a number of drilling platforms in the North Sea is forming a high speed rescue unit to cope with emergency situations which may occur. The rescue unit comprises of 6 personnel, who, for reasons of flexibility, undergoes the same comprehensive training programme. The six personnel are assessed as to their suitability for various specialist tasks and the marks they received in the training programme are given in the Table:

	Trainee No.					
Specialist Task	*I*	*II*	*III*	*IV*	*V*	*VI*
Unit Leader	21	5	21	15	15	28
Helicopter Pilot	30	11	16	8	16	14
First Aid	28	2	11	16	25	25
Drilling Technology	19	16	17	16	19	8
Fire Fighting	26	21	22	28	29	24
Communications	3	21	21	11	26	26

Based on the marks awarded, what role should each of the trainee be given in the rescue?

9. A project work consists of four major jobs for which four contractors have submitted tenders. The tender amount quoted in thousands of rupees are given in the matrix as:

	Jobs			
Contractors	J_1	J_2	J_3	J_4
C_1	15	29	35	20
C_2	21	27	33	17
C_3	17	25	37	15
C_4	14	31	39	21

Find the assignment which minimises total cost of the project. Each contractor has to be assigned one job.

10. A solicitors firm employs typists on hourly-piece-rate basis for their daily work. There are five typists for service and their charges and speeds are different. According to an earlier understanding, only one job is given to one typist and the typist is paid for full hour even if he works for a fraction of an hour. Find the least cost allocation for the following data :

Typist	*Rate per hrs.* (₹)	*No. of pages typed per hour*	*Job*	*No. of Pages*
A	5	12	P	199
B	6	14	Q	175
C	3	8	R	145
D	4	10	S	298
E	4	11	T	178

11. A city corporation has decided to carry out road repairs on four main arteries of the city. The government has agreed to make a special grant of ₹ 50 lakhs towards the costs with a condition that the repairs must be done at the lowest cost and quickest time. If conditions warrant, then a supplementary token grant will be considered favourably. The corporation has floated tenders and 5 contractors have sent in their bids. In order to expedite work, one road will be awarded to only one contractor.

	Cost of Repairs (₹ lakhs)			
Contractors/Road	R_1	R_2	R_3	R_4
C_1	9	14	19	15
C_2	7	17	20	19
C_3	9	18	21	18
C_4	10	12	18	19
C_5	10	15	21	16

(*i*) Find the best way of assigning the repair work to the contractors and the costs.

(*ii*) If it is necessary to seek supplementary grant, then what would be the amount sought?

(*iii*) Which of the five contractors will be unsuccessful in his bid ?

12. Six wagons are available at six stations A, B, C, D, E and F. These are required at stations I, II, III, IV, V and VI. The distances (in kms) between various stations are given by the following table.

	I	II	III	IV	V	VI
A	20	23	18	10	16	20
B	50	20	17	16	15	11
C	60	30	40	55	8	7
D	6	7	10	20	100	9
E	18	19	28	17	60	70
F	9	10	20	30	40	55

How should the wagons be transported in order to minimise the total kms covered?

13. Solve the following assignment problem for minimum cost. The cost structure is given below:

	M_1	M_2	M_3	M_4	M_5	M_6
T_1	31	62	29	42	15	40
T_2	12	20	40	55	70	38
T_3	15	30	50	42	20	22
T_4	35	40	37	44	27	33
T_5	20	30	28	16	32	23
T_6	65	30	32	50	42	20

14. At the end of the cycle of schedules, a trucking firm has a surplus of one vehicle in each of the cities 1, 2, 3, 4 and 5 and a deficit of one vehicle in each of the cities A, B, C, D, E and F. The costs (in rupees) of transporting and handling between the cities with a surplus and the cities with a deficit are shown in the following table :

	To City					
From City	*A*	*B*	*C*	*D*	*E*	*F*
1	134	116	167	233	194	97
2	114	195	260	166	178	130
3	129	117	48	94	66	101
4	71	156	92	143	114	136
5	97	134	125	83	142	118

Find the assignment of surplus vehicles to deficit cities that will result in a minimum total cost. Which city will not receive a vehicle?

15. A salesman has to visit four cities A, B, C and D. The distances (in hundred kms) between the four cities are as follows :

	To			
From	*A*	*B*	*C*	*D*
A	—	4	7	3
B	4	—	6	3
C	7	6	—	7
D	3	3	7	—

If the salesman starts from city A and has to come back to city A, which route should he select so that total distance travelled by him is minimised?

16. Six salesmen are to be allocated to six sales regions so that the cost of allocation of the job will be minimum. Each salesman is capable of doing the job at different costs in each region. The cost (in rupees) matrix is given below :

	Salesmen					
Regions	*I*	*II*	*III*	*IV*	*V*	*VI*
A	15	35	0	25	10	45
B	40	5	45	20	15	20
C	25	60	10	65	25	10
D	25	20	35	10	25	60
E	30	70	40	5	40	50
F	10	25	30	40	50	15

(*i*) Find the allocation to give minimum cost. What is the minimum cost?

(*ii*) If the figures given in the above table represent the earning of each salesman at each region, then find an allocation so that the earning will be maximum. Also work out this maximum possible earning.

17. Five different machines can process any of the five required jobs as follows with different profits resulting from each assignment:

	Machines				
Jobs	*A*	*B*	*C*	*D*	*E*
1	30	7	40	28	40
2	40	32	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	39	32	41	34	39

Find out the maximal profit possible through optimal assignment.

18. A sales manager has to assign salesmen to four territories, He has four candidates of varying experience and capabilities and assess the possible profit for each salesman in each territory as given below. Find the assignment which maximises profit.

Salesmen	*Territories*			
	A	*B*	*C*	*D*
1	35	27	28	37
2	28	34	29	40
3	35	24	32	33
4	24	32	25	82

19. A computer centre has got three expert programmers. The centre needs three application programmes to be developed. The head of the computer centre, after studying carefully experts to the application programmes, estimates the computer time (in minutes) required by the experts to the application programmes as follows :

	Programmes		
Programmers	*A*	*B*	*C*
1	120	100	80
2	80	90	110
3	110	140	120

Assign the programmers to the programmes in such a way that the total computer time is the least.

20. A company is considering an expansion into five new sales territories. The company has recruited four new salesmen. Based on the salesman's experiences and personality traits, the sales manager has assigned ratings to each of the salesmen for each of the sales territories. The ratings are as follows:

	Territory				
Salesman	*1*	*2*	*3*	*4*	*5*
A	75	80	85	70	90
B	91	71	82	75	85
C	78	90	85	80	80
D	65	75	88	85	90

Suggest optimal assignment of the salesmen. If for some reason, salesman D cannot be assigned to territory 3, will the optimal assignment be different? If so, what would be the new assignment schedule?

9.12 Summary

This unit is summarised by using these points:

- **Assignment Problem:** A Special case of linear programming model dealing with the allotment or assignment of given assets to its optimal place.
- **Prohibitive Assignment:** When a particular allocation or assignment is not feasible or permitted due to technical or administrative reasons.
- **Optimal Solution:** When it is possible to allocate resources to their unique place for optimal cost/profit/distance.

9.13 Key Words/Abbreviations

- **Hungarian Assignment Method (HAM):** The Hungarian method is a combinatorial optimization algorithm that solves the assignment problem in polynomial time.
- **Unbalancing and Prohibitive Assignment:** Unbalanced Assignment problem is an assignment problem where the number of facilities is not equal to the number of jobs.

- **Multiple Optimal Solution:** The multi-resource generalized assignment problem is encountered when a set of tasks have to be assigned to a set of agents in a way that permits assignment of multiple tasks to an agent subject to the availability of a set of multiple resources consumed by that agent.

9.14 Learning Activity

1. Explain the following terms

   (i) Regret matrix

   ____

   ____

   (ii) Prohibited problem

   ____

   ____

   (iii) Dummy

   ____

   ____

9.15 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. How can you call the Assignment problem as a special case of Transportation problem?
2. How do you balance the assignment matrix? How is it different from that for transportation problem?
3. Discuss various characteristic of an assignment problem?

4. Why do you use only unique zero elements for assignment in Hungarian Method?

5. Explain the allocations for multiple optimal solution situations in Hungarian Method?

6. Why is Travelling Salesman problem so unique the normal assignment situation?

B. Multiple Choice/Objective Type Questions

1. Assignment Problem is basically a ______________________.

   (a) Maximisation Problem (b) Minimisation Problem

   (c) Transportation Problem (d) Primal problem

2. The horizontal and vertical lines drawn to cover all zeros of total opportunity matrix must be

   (a) Equal to each other

   (b) Equal to $m \times n$ (where m and n are number of rows and columns)

   (c) $m + n$ (where m and n are number of rows and columns)

   (d) Number of rows or columns

3. To balance the assignment matrix we have to ___________________.

   (a) add a Dummy row

   (b) add a Dummy column

   (c) add either a dummy row or column depending on the situation

   (d) You cannot balance the assignment matrix

4. The total opportunity cost matrix is obtained by doing _______________.

   (a) Row operation on row opportunity cost matrix

   (b) Column operation on row opportunity cost matrix

   (c) Column operation on column opportunity cost matrix

   (d) None of the above

5. The assignment problem will have alternate solutions __________________.
   (a) when total opportunity cost matrix has at least one zero in each row and column
   (b) When all rows have two zeros
   (c) When there is a tie between zero opportunity cost cells,
   (d) If two diagonal elements are zeros

Answers:

1. (b), 2. (d), 3. (c), 4. (b), 5. (c)

9.16 References

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UNIT 10 PERT AND CPM

Structure:

10.0 Learning Objectives

After studying this unit, you will be able to:

- Describe the use of Networks for Resources Management in a Project.
- Define the methodology of Network Analysis.
- Differentiate the Objectives/Advantages of the Network Analysis.
- Discuss the basic Rules to Draw Network.
- Obtain the Time Estimates for the Project Activities
- Define the concept of Slacks and Floats.
- Determining the Critical Path.
- Difference between CPM and PERT Networks.
- Discuss thw time and Cost Analysis through Networks.
- Illustrate the concepts through Solved Examples.
- Analyse yourself through Self Assessment Problems

10.1 Introduction

Any management system revolves around utilisation of human as well as non-human resources. It is generally observed that for carrying out management function, the resources are at premium. Hence an efficient manager is one who optimises his inputs to achieve the best and gets maximum out of his resources under most trying conditions. While motivation of human resources plays a very major role in accomplishing the tasks by a given schedule, the optimisation of the non-human assets for varied and large number of activities, therefore, needs certain tried out techniques to be applied.

Completion of any project by a certain definite time schedule is the essence of any management challenge. The organisation has to decide the goal or the mission of the project in very clear terms and then plan the work by working out resources for its completion. The scheduling of the activities will be based on three important constraints.

(a) Time schedule.

(b) Money constraint.

(c) Manpower and equipment constraints.

Within the above constraints, the detailed planning of the project must be carried out based on

1. Mission of the project/objective of the management.
2. Extent of control desired/critically of the project.
3. Resources and techniques available for control.

Network techniques are the pictorial representation of the activities and their interrelationship to help in the planning, scheduling and controlling the project. In simple and small level project, there may not be a requirement of use of very sophisticated techniques, but when project is very large and there are very complex activity relationship with resources being very limited, network techniques come to the help of the project manager in a big way.

Though there are quite a few such techniques available today along with effective software packages like MS-project and Primavera, we shall restrict our discussion to the basic technique of CPM (Critical Path Control) and PERT (Programme Evaluation and Review Technique). Though most commonly used and most easily understood techniques of Bar-Chart and Gantt-Chart are still in use for small projects, large projects need detailed planning and control, thereby needing the use of CPM and PERT with newer and latest softwares mentioned above.

CPM and PERT were developed in late 1950's, though quite independently, but with the same purpose and using the same terminology. The minor variation is that PERT was used for dealing with uncertainties in activity completion time. The major strength of CPM was its ability to take care of the trade-off facility in terms of time and cost variations *i.e.*, it caters for additional resources for counteracting time over-run and *vice-versa*. In the present context, the difference between CPM and PERT has largely vanished.

Widely diverse projects are amenable to analysis by PERT and CPM. Few of them are listed below:

1. Research and development programme
2. Construction of a plant
3. Building a mega project of irrigation
4. Launching of a space-ship
5. Over-haul of an organisation
6. Training of manpower
7. Starting an adult literacy programme
8. Arranging a dinner/cocktail party

10.2 Network Technique for Planning

Planning is basically a process of working out specific number and types of activities with their associated time schedules. The work is to be logically and methodically structured into a step by-step planning framework, where CPM/PERT are used extensively. The methodology of planning leading to the use of these techniques can be described as below.

Project Identification and Definition—Analysis of job is an initial step of planning work. This would include the determination of set of activities and their sequence of performance for proper implementation.

Resources Planning—Based on the quantum of work under each activity, the resources need be calculated in terms of personnel, equipment, time, cost, materials etc. specifying level of skills, type and efficiency of the equipment, time schedule with reference to inter-relationship of various activities, the quantum and schedule of availability of money required and the details of materials as per Master work schedule.

Project Scheduling—We now work out a detailed layout of the activities with specific time schedule.

Project Control—Control methodology by the use of Network system is a must for monitoring the progress of work in terms of its physical and financial set up. Alternate plans of 'what-if' analysis must also be prepared by using Network system extensively.

10.3 Objectives of the Network Analysis

The use of Network analysis is made in achieving following objectives—

1. Minimisation of total project cost.
2. Minimisation of total project time.
3. Trade-off between the time and cost of the project.
4. Optimisation of human and non-human resources.
5. Minimisation of conflicts, delays and interruptions.

10.4 Advantages of Network Analysis

Network Techniques are handy tools for a project manager to achieve the following advantages:

Planning Stage—listing out activities and then sequencing, resources planning and estimation of cost and time for various activities. This helps in defining the total project.

Scheduling Stage—working out inter-relationships amongst various activities, their inter-dependence and possible improvements, scheduling of flow path of activities and associated resources. This would indicate the largest schedule called critical path of the project. This helps in getting the quantum of optimal resources.

Controlling Stage—Having used Network techniques extensively for planning and scheduling of the project, these become effective tools for monitoring and controlling the time and cost schedules. Constant review, bringing status report into focus, can help in reallocation of resources wherever bottlenecks are noticed. Trade-off between time, cost and environmental conditions can be achieved effectively.

10.5 Basic Rules of Network Analysis

Basically CPM or PERT use a graphical presentation depicting the project in its entirety. Networks can be drawn for different levels of project such as project level, sub-project level, tasks and activities levels etc. While drawing out these graphical forms of Network, some conventions or basic rules have to be followed. The rules for establishing the Network system are enumerated below :

1. A decision-maker prepares the list of activities and their inter-dependence and inter-relationships. Each activity is represented by a circle called node or event and an arrow, i.e., one activity should have only one arrow representing it as given in Fig. 10.1 below :

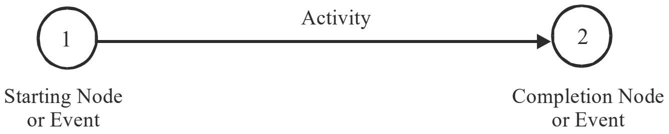


Fig. 10.1

It assures that time flows in the forward direction, but the length of an arrow has no significance as to decide proportionality of time.

2. Each activity must have a preceding and a succeeding event. Hence an activity is designated by a pair of preceding and succeeding events so numerically numbered. The head-events always should have a number higher than that of the tail event. Starting event is the preceding event and completion event is the succeeding event, as 1 and 2 above (Fig. 10.1).

3. There should be no loops in the project network. The network as given below is NOT permissible.

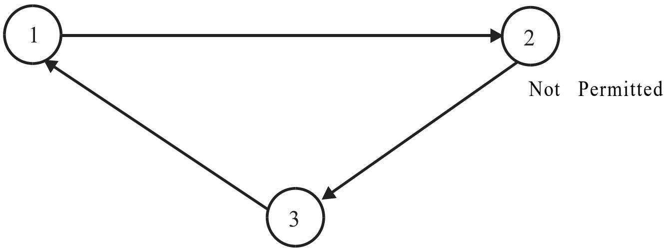


Fig. 10.2

4. There cannot be more than one activity having the same preceding and succeeding events. The following is NOT permitted.

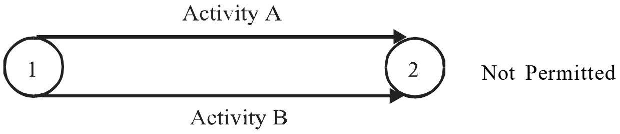


Fig. 10.3

5. The same event can be a preceding or a succeeding activity to more than one activity of the network. This shows the precedence of operations of the project. Numbering of events should be in the order of happening i.e., succeeding event should be numbered only when all the preceding events have been numbered. Generally beginning of the project should be denoted by starting node (single) and end by a single completion node. A situation can be illustrated like this. Numbering of events have been suggested by Fulkerson's Rule as given above.

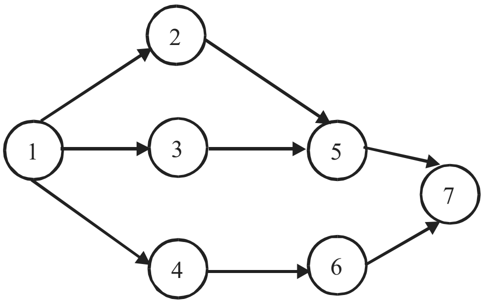


Fig. 10.4

6. There can be some activities happening simultaneously or concurrently and these are called concurrent activities. In order to establish proper relationship, we use the concept of dummy activity. Dummy activity does not consume time or any other input resources. In Fig. 10.5, activity 1-2 and 3-4 are concurrent activities, but dummy activity 4-2 means that before we undertake activity 2-5, activity 1-2 and 3-4 both should get completed, whereas 4-6 can be undertaken without any reference to 2-5. Similarly in Fig. 10.6, activity 1-2 and 1-3 are concurrent activities. Activity 3-4 can be performed without any relevance of 1-2 being complete or not. But before taking up activity 4-5, activities, 1-2 and 3-4 both should get completed. Thus dummy activity has shown relationships of sequence of such connected activities.

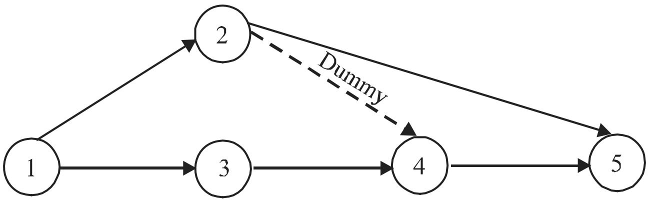


Fig. 10.5

Now, various steps to draw a network can be explained by taking an appropriate example and developing a network.

10.6 Obtaining Time Estimate

For any job to be performed, it has to be broken down into activities and then time estimation for each activity needs be worked out. For this purpose, we estimate the activity time in the following manner—

t_o = optimistic time

t_m = most likely time

t_p = pessimistic time

t_e = weighted average time
or expected time for completion

Optimistic time : It is the shortest time that may be required for completion of an activity, when all resources are available and no bottleneck is expected.

Most likely time : It is the amount of time required to complete an activity under real-life situation where there may be some bottlenecks experienced but quickly resolved without losing time.

Pessimistic time : It is the longest time required for completion of an activity when unforeseen or unexpected interruptions occur during its completion.

Then expected time = Weighted average time

$$\therefore \qquad t_e = \frac{t_{\text{o}} + 4\ t_m + t_p}{6}$$

In case the time estimates described above are not deterministic in nature, the uncertainty is taken care of by calculating standard deviation (SD or s) or the variance (V) of the duration t_e. This is expressed and denoted in a PERT network, where it is assured that such activity time follows Beta Distribution with the following characteristics.

(*a*) It is a uni-nodal distribution.

(*b*) It has finite non-negative end points.

(*c*) It is not necessarily symmetrical about nodal value.

The relationships for estimated mean time, standard deviation and variance are given below—

$$t_e = \frac{t_o + 4\ t_m + t_p}{6}$$

$$s_{te} = \left(\frac{t_p - t_o}{6}\right)$$

and $$V_{te} = s_{te}^{\ 2}$$

To obtain time estimates, following aspects be borne in mind:

1. Time estimates should be obtained for each activity independently on the network, but not along a specific path.
2. The time estimates t_o, t_m, t_p should be obtained independently of each other. but total time required for the project should not cloud up time estimation of related activities.
3. Time estimates t_o, t_m and t_p to be treated as estimates only for planning the work and not as committed schedule.
4. Estimates should include permissible allowances and environmental occurrences.

The concept has been illustrated in problem 10.2.

10.7 Calculations for EOT and LOT

To depict such calculated Earliest Occurrence Time (EOT) and Latest Occurrence Time (LOT) for various events, the event nodal circle is divided into 3 parts. Top half indicates the event serial, whereas the lower half is divided into 2 parts, the left indicating the EOT and the right one mentioning the LOT described as follows.

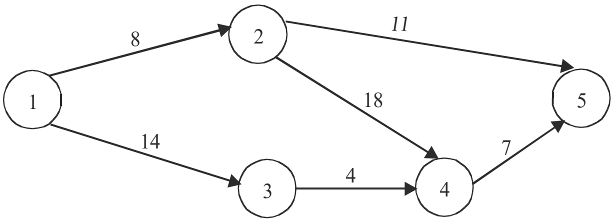


Fig. 10.6

Determination of the Earliest Occurrence Time (EOT) : We consider an event to occur, when all activities leading to that even have been accomplished. Thus from the given network diagram, event 4 will occur only when activities (1-2), (1-3), (2-4) and (3-4) have been completed. Since activity (2-4) cannot start unless event 2 has occurred, the EOT for event 2 is 8 units. Likewise activity (3-4) can not begin unless event 3 has occurred. This activity (1-3) takes 14 units of time for completion. Hence event 4 would occur only when both the paths 1-2-4 and 1-3-4 have been completed. The Earliest Occurrence Time (EOT) of an event is the time when this event can occur the earliest.

Path 1-2-4 takes 26 units (8 + 18) of time whereas Path 1-3-4 takes only 18 units (14 + 3) of time. Thus EOT for event 4 will be 26 units (8 + 18) of time, considering beginning of the event 1 as zero.

The EOTs so calculated can be represented on the network as follows :

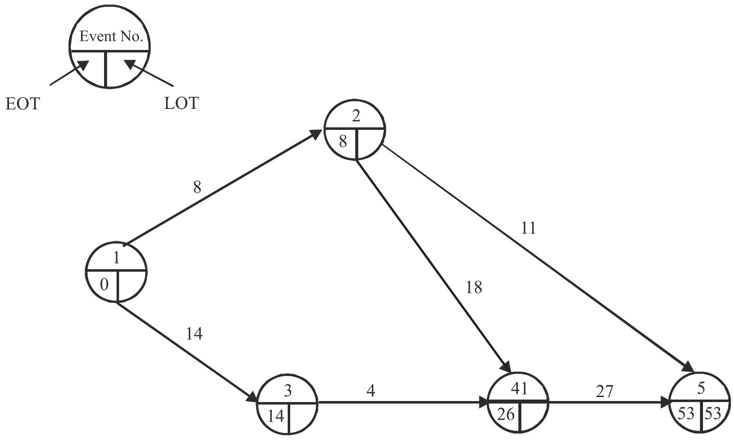


Fig. 10.7

It can be seen that the EOT of the end event (event 5 in this case) would represent the minimum time required for completion of the project. With the assumption that the next activity would start soon after the occurrence of the event preceding it, these times would also mean the earliest starting time (EST) of the next activity or succeeding activity. This is called Forward pass in CPM.

Determination of Latest Occurrence Time (LOT) : The latest occurrence time for any event represents the latest time by which time the event can occur, based on the time permitted for the project by a definite period. Hence calculation of the latest occurrence time (LOT) starts from the end event, whose EOT and LOT are set at the same level. In the configuration given above, the LOT for event 5 will be 53 units of time.

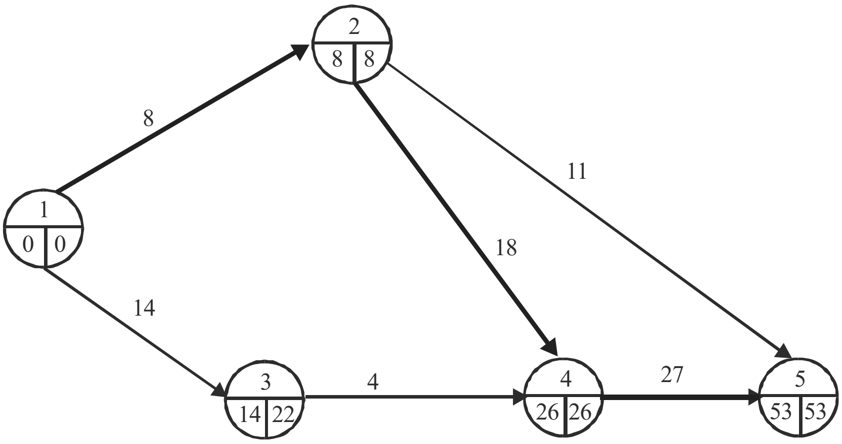


Fig. 10.8

Working backwards (called backward pass) LOT for event 5 will, therefore, be 53 units and for event 4, it will be LOT of end event minus the duration of activity (4-5). LOT for event 4 works out to 26. Similarly LOT for event 2 will be 53 - 10 = 43 units working on path 2-5, but it is connected with 5 through (2-4) and (4-5). Since event 4 is connected with event 2 also, event 4 can finish latest by 43 - 18 = 25 units also, but it is earlier than 26 units calculated from activity (4-5). Hence LOT for event 4 will be 26 units only. Similarly we calculate LOT for other events also and these are indicated in bottom right half of the event node, as follows:

From the above description and values of EOT and LOT marked on the network diagram, we find that events 1, 2, 4, 5 have the same EOT and LOT. Thus there is no slack on these events. Such a path i.e., 1-2-4-5 is called critical path, since there is no cushion available for activity starting and completion. Wherever these figures are different, the cushion available between EOT and LOT is called the slack for the event. Path 1-2-4-5 will also be the longest path of the network.

10.8 Slacks

It is the time available as cushion on a node and is the difference of LOT and EOT. Slacks for the above network diagram are given below:

Event	*LOT*	*EOT*	*Slack = LOT – EOT*
5	53	53	0
4	26	26	0
3	22	14	8
2	8	8	0
1	0	0	0

10.9 Determination of the Critical Path

After having obtained the estimated expected time of each activity, we can now work out the time schedule for reaching a particular event or the total time required for the project. From the concept of the total elapsed time for a particular event to occur, it is then easy to determine the critical path on the network based on the longest time schedule for all activities.

Thus two conditions must be satisfied for critical path.

1. It is the path joining nodes of zero slacks.
2. It is also the largest path of the network.

10.10 Computation of Activity Floats

There are three measures of float : (*a*) total float (*b*) free float and (*c*) independent float. From the given estimates of activity time and event slacks, these measures can be worked out as follows:

Total Float : Of an activity is the extra period available for completion of the activity, if the activity is started as early as possible. Total float of an activity is equal to

Latest Occurrence Time for succeeding activity - Earliest Occurrence Time for the activity - Duration of the activity.

or, Total Float = $LOT_{(S)} - EOT_{(A)} - T_A$

Free Float : Of an activity is the extra time available to complete the activity when activity start on LOT of the preceding event and is completed by the EOT of succeeding event.

Free Float = $EOT_{(S)} - EOT_{(A)} - T_A$

Independent Float : Of an activity is the extra time available to complete the activity when activity starts at EOT of the preceding event and completed by EOT of its succeeding event.

$\therefore$ Independent Float = $EOT_{(S)} - LOT_{(A)} - T_A$

This can be indicated for activity (i, j) as follows

Total Float = $TF(i, j) = LOT(j) - EOT(i) - d(ij)$

Free Float = $FF(i, j) = EOT(j) - EOT(i) - d(ij)$

Independent float = $IF\ (ij) = EOT\ (j) - LOT\ (i) - d(ij)$

Where $d(ij)$ is the duration of activity (i, j).

A table given below will illustrate the working out of floats

Activity	*Duration*	*EST (i)*	*EFT (j)*	*LST (i)*	*LFT (j)*	*TF*	*FF*	*IF*
1-2	13	0	13	0	13	0	0	0
1-3	12	0	12	6	18	6	0	0
2-4	2	13	15	24	26	11	5	5
3-4	8	12	20	18	26	6	0	(6)
2-5	15	13	28	13	28	0	0	0
4-5	2	20	22	26	28	6	6	0

It can be seen that activities, which do not have a float even under most favourable conditions, are critical for the project and hence would form a part of critical path. Here activities (1-2) and (2-5) are critical activities.

10.11 PERT Model

For drawing networks so far, we have used the concept of expected mean duration estimates of activities. This was obtained based on optimistic (t_o), pessimistic (t_p) and the most likely time (t_m). In real life situation, there are bound to be variabilities in the project duration and this is considered in formulating the PERT model.

The variability in the PERT method can be measured either by variance or by its square root, the standard deviation as given in the problem 2. The method followed for calculating the standard deviation of the duration is as follows:

1. Determine the standard deviation of the duration of each activity on the critical path.
2. Determine the standard deviation of the total duration of the project on critical path with the help of all individual activity standard deviations.
3. Variance of the project can be, then, calculated by squaring the standard deviation so obtained.

The standard deviation of each activity can be calculated by using the relationship.

$$\sigma = \left(\frac{t_p - t_o}{6}\right)$$

where t_p = Most pessimistic time

t_o = Most optimistic time

and σ = Standard deviation of the activity

Project Variance $V = \sqrt{\sigma_{c_1}^2 + \sigma_{c_2}^2 + \sigma_{c_3}^2 + \ldots\ldots.}$

where σ_{c1} = S.d. of critical activity 1 and so on

It can be illustrated as follows:

Activity	t_p	t_o	$\sigma = \frac{t_p - t_o}{6}$	$V = \sigma^2$
1 - 2	20	8	2	4
2 - 5	12	4	1.67	2.8

Variance of the project (critical path duration) = Sum of activity variances

and standard deviation of the critical path duration = $\sqrt{\sum V}$

In the above example, SD of the critical path = $\sqrt{4+2.8}$

= 2.6

Since for a long project, there will be a large number of activities on the critical path, critical path duration can be assumed to be following a normal distribution and hence the probabilities of values lying within a certain range will be

Mean ± σ, Mean ± 2σ, Mean ± 3σ with respective probabilities as 0.682, 0.954 and 0.998, which are obtained by calculating the areas under the curve for these variation limits.

When mean time T, the probability of the completion of the project by certain specified date (D) can be calculated

$$Z = \frac{D-T}{\sigma}$$

Then cumulative probabilities upto *Z* can be obtained from the probability distribution table for normal distribution.

By this method, probability of completion of the project by a certain time schedule can be estimated. These probabilities of each individual activity can be written along with mean time duration to depict the PERT network (say activity can have duration probability marked on arrow).

10.12 Time Cost Analysis

Whereas PERT model was primarily developed for project with uncertainty of events and time, CPM was developed for deterministic approach. Following assumptions are made while dealing with CPM analysis:

1. Project costs have two distinct sub-components; the direct costs consisting of direct labour and material costs and the indirect costs involving items like indirect supplies, rent, taxes, insurance and overhead services etc.
2. Certain activities of the project can be expedited by crashing the time schedule with the help of additional resources.
3. Crashing of time i.e., time reduction enhances direct costs like overtime, extra wages and emergency costs of services etc. Based on these assumptions, let us consider crashing the time schedule of the project. The procedure followed is as under :
   (*a*) Obtain critical path in the network by normal estimated mean times.
   (*b*) Examine the cost-time slope of the activities on the critical path and start crashing of activity with the least slope.
   (*c*) Construct the new critical path after crashing of an activity has been completed and revised schedule and costs obtained.
   (*d*) Repeat steps (*ii*) and (*iii*) till crashing of all the activities on such revised critical path at each iteration has been obtained.

Solved Examples

Problem 1 :

All activities of a project are given below along with their inter-relationships. Draw the network for the same

Activity	*Description of the Activity*	*Predecessor Activity*
A	Mobilisation of resources	—
B	Erection of dam site	A
C	Excavation	B
D	Lay foundation	C
E	Erect form work	D
F	Erect columns/Walls	E
G	Draw electric wiring	F
H	Install plumbing	F

I	Plaster walls	F
J	Erect interior work	G, H, I
K	Plaster outside	H

Solution:

This network can be drawn as Fig. 10.9 below

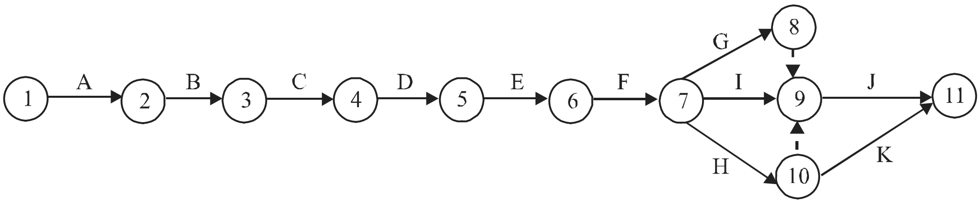


Fig. 10.9

Problem 2:

For the network given below, work out the expected time of completion of various activities, with the given data.

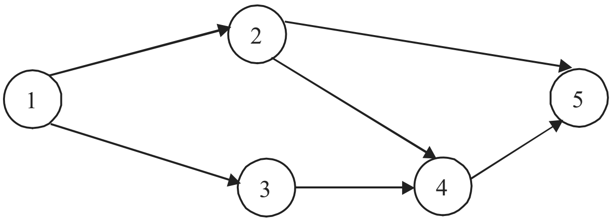


Fig. 10.10

Activity	*Optimistic Time*	*Pessimistic Time*	*Most Likely Time*
1 - 2	5	12	7
1 - 3	12	17	13
2 - 4	15	21	18
3 - 4	2	5	3
2 - 5	8	14	10
4 - 5	21	35	26

Solution :

For working out the expected estimated average time for all the activities, the table can be organised as follows:

Acitvity	t_o	t_p	t_m	t_e
1 - 2	5	12	7	7.5 say 8
1 - 3	12	17	13	13.5 say 14
2 - 4	15	21	18	18
3 - 4	2	5	3	3.1 say 4
2 - 5	8	14	10	10.1 say 11
4 - 5	21	35	26	26.7 say 27

This can be described on the network as follows (Normally t_e is rounded off to the next higher integer value).

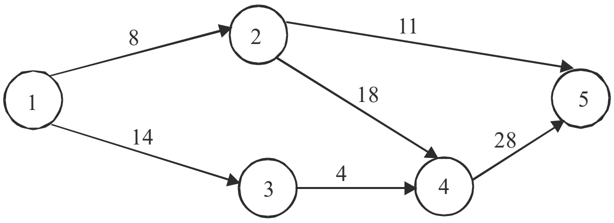


Fig. 10.11

The same time estimates can be used for the PERT depiction, by working out the variance for individual activities.

Activity	t_o	t_p	t_m	t_e	*Variance* $[(t_p - t_o)/6]^2$
1 - 2	5	12	7	8	1.22
1 - 3	12	17	13	14	0.7
2 - 4	15	21	18	18	1.0
3 - 4	2	5	3	4	0.25
2 - 5	8	14	10	11	1.0
4 - 5	21	35	26	27	5.29

Problem 3 :

Obtain the critical path and project duration for the following PERT network.

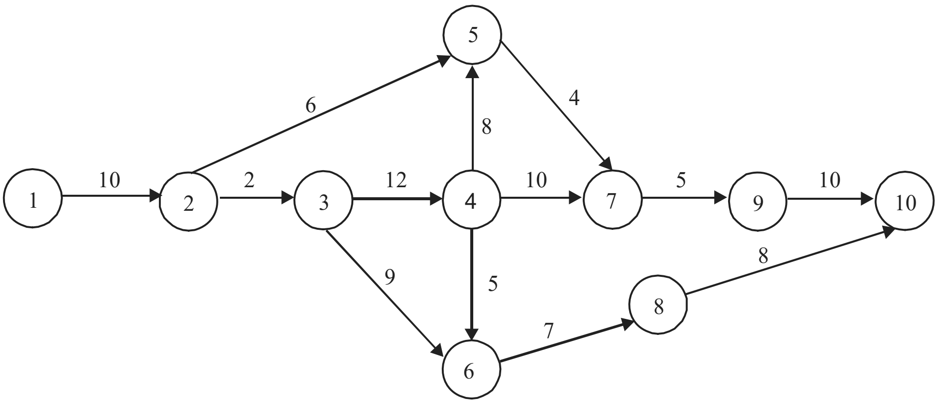


Fig. 10.12

Solution :

Starting from event 1 on forward pass, we get the EOTs for various events as follows:

EOT for event 1 = 0

(Start event)

EOT for event 2 = 10 (activity 1-2 takes 10 units of time)

EOT for event 3 = 10 + 2 = 12

EOT for event 4 = 12 + 12 = 24

EOT for event 5 = Max [10 + 6, 24 + 8] = 32

EOT for event 6 = Max [12 + 9, 24 + 5] = 29

EOT for event 7 = Max [24 + 10, 32 + 4] = 36

EOT for event 8 = 29 + 7 = 36

EOT for event 9 = 36 + 5 = 41

EOT for event 10 = Max [36 + 8, 41 + 10] = 51

Similarly, going on the backward pass, we can calculate LOT for various events, starting from event 10 where EOT = LOT = 51.

LOT for event 10	= 51
LOT for event 9	= 51 - 10 = 41
LOT for event 8	= 51 - 8 = 43
LOT for event 7	= 41 - 5 = 36
LOT for event 6	= 43 - 7 = 36
LOT for event 5	= 36 - 4 = 32
LOT for event 4	= Min [36 – 5, 36 – 10, 32 – 8]
	= 24
LOT for event 3	= Min [24 – 12, 36 – 9]
	= 12
LOT for event 2	= Min [12 – 2, 32 – 6]
	= 10
LOT for event 1	= 10 – 10 = 0

The graphical representation is given in Fig. 10.13.

The critical path is 1 - 2 - 3 - 4 - 5 - 7 - 9 - 10 with project duration of 51 units of time. (Refer Fig. 10.13).

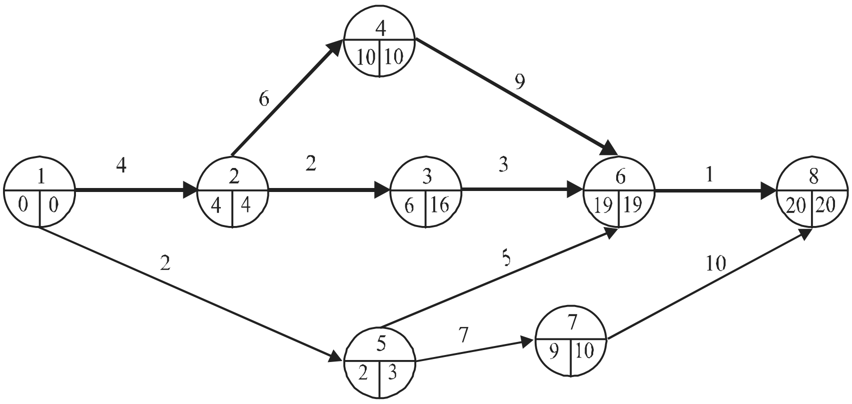


Fig. 10.13

Problem 4 :

Consider the following schedule of activities and related information for the construction of a new plant.

Activity	*Expected Time*		*Expected Cost (₹ 00,000's)*
	Months	*Variance*	
1 - 2	4	1	5
2 - 3	2	1	3
3 - 6	3	1	4
2 - 4	6	2	9
1 - 5	2	1	2
5 - 6	5	1	12
4 - 6	9	5	20
5 - 7	7	8	7
7 - 8	10	16	14
6 - 8	1	1	4

You should assume that the cost and time required for one activity are not dependent upon the cost and time of any other activity and variations are expected to follow a normal distribution. You are required to calculate :

(*a*) the critical path

(*b*) expected cost of construction of the plant

(*c*) expected time required to build the plant

(*d*) the standard deviation of the expected time.

Solution :

The network diagram of the project activities is drawn as follows. It reflects EOT, LOT, Critical path and the project duration.

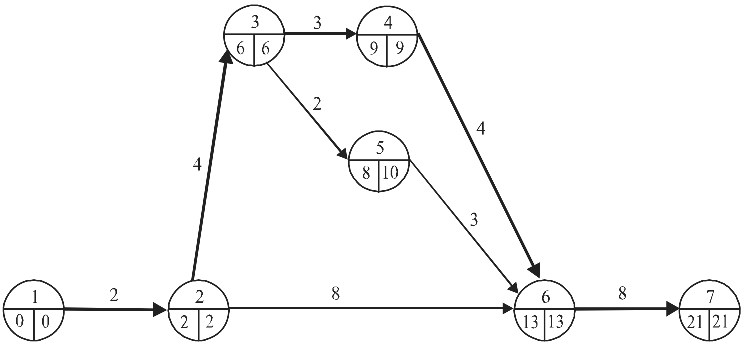


Fig. 10.14

(*a*) Critical path drawn in thick line is 1 - 2 - 4 - 6 - 8. Since there are no slacks on these activities and it is the longest path.

(*b*) Expected cost of construction = 5 + 3 + 4 + 9 + 2 + 12 + 20 + 7 + 14 + 4 = ₹ 80,00,000.

(*c*) Expected time of completion = EOT or LOT of the end event.

= 20 months

(*d*) Standard deviation of the expected time obtained as 20 months is

Activity	*Variance*
1 - 2	1
2 - 4	2
4 - 6	5
6 - 8	1
Total	**9 months**

Standard deviation of the expected time $= \sqrt{9} = 3$ months

Problem 5 :

The activities, duration and direct activity costs are given below. The indirect cost is ₹ 3,000 per week. Obtain the crash cost and duration of the project.

Activity	*Time in Weeks*		*Cost*		*Cost to Expedite per week (cost slope)*
	Normal	*Crash*	*Normal*	*Crash*	
1 - 2	2	2	3000	3000	—
2 - 3	4	3	4000	5000	1000
2 - 6	8	8	6000	6000	—
3 - 4	3	2	2000	3500	1500
3 - 5	2	2	2000	2000	—
4 - 6	4	3	4000	5000	1000
5 - 6	3	3	4000	4000	—
6 - 7	8	5	8000	12000	1333

Solution :

Step 1 : Drawing the network, the critical path has been obtained.

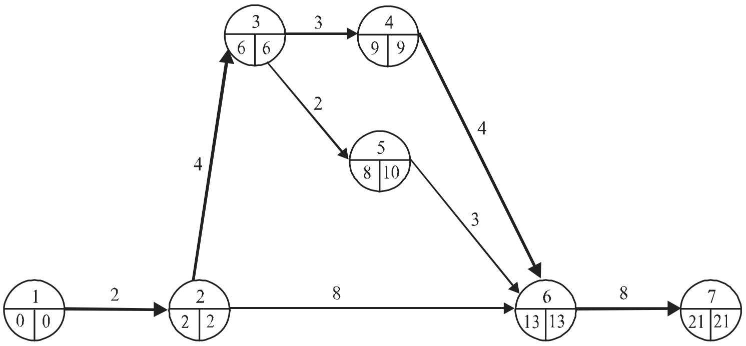


Fig. 10.15

Hence Critical Path is 1-2-3-4-6-7 (Fig. 10.15).

Step 2 : The least cost-time slope of activities on the critical path is 1,000 on activity 2-3 an well as 4-6. Hence any of the two activities can be taken up for crashing. Selecting 4-6 we get the network as follows: (Refer Fig. 10.16)

Thus the Critical Path remains as 1-2-3-4-6-7 with crashing of activity 4-6 from 4 weeks to 3 weeks and project duration reducing from 21 to 20 weeks.

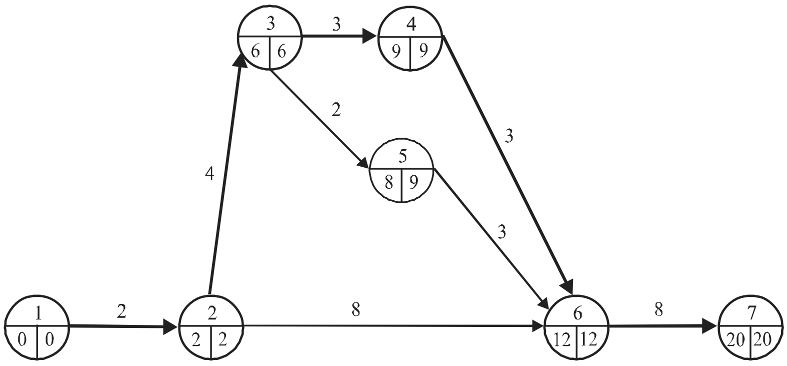


Fig. 10.16

Step 3 : Since critical path remains the same, we can now select activity 2-3 for crashing, which can be crashed from 4 to 3 weeks.

New network can be drawn as follows :

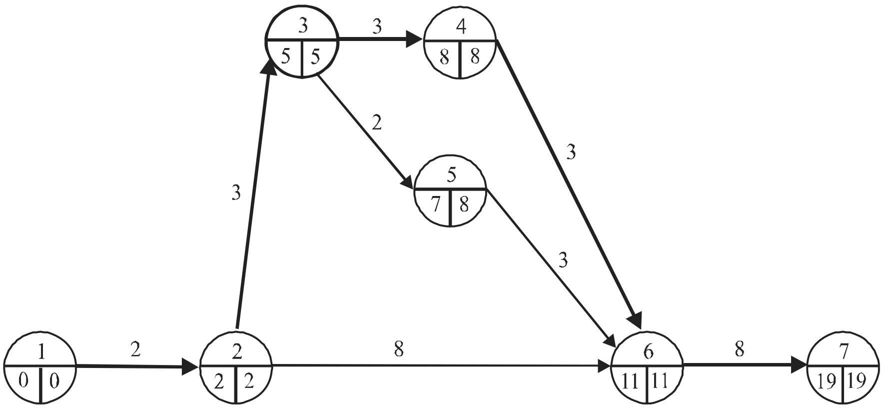


Fig. 10.17

Revised Critical Path is 1-2-3-4-6-7 which is same as above with total duration reducing from 20 to 19 weeks (Refer Fig. 10.17).

Step 4 : Now we can take up activity 6-7 (next least slope i.e., 1,333) for crashing from 8 weeks to 5 weeks giving network as follows : (Refer Fig. 10.18)

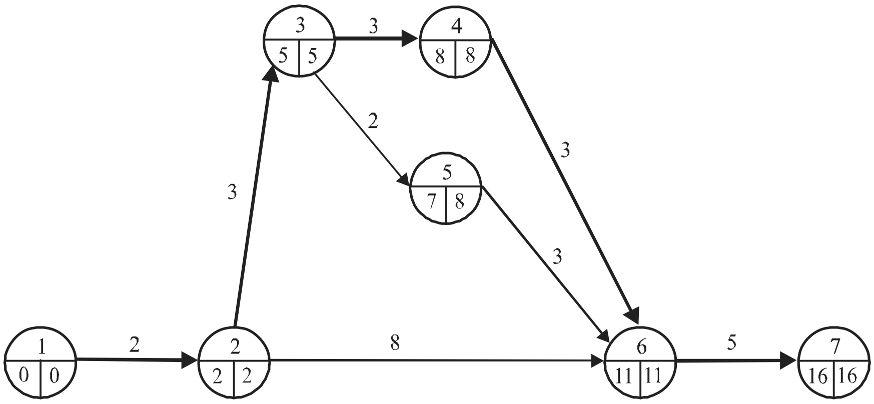


Fig. 10.18

Step 5 : Critical Path still remains 1-2-3-4-6-7 with revised duration of 16 weeks. Next activity qualified for crashing on Critical Path is 3-4 with next slope level of 1,500. This activity is to be crashed from 3 to 2 weeks and revised network can be drawn as follows :

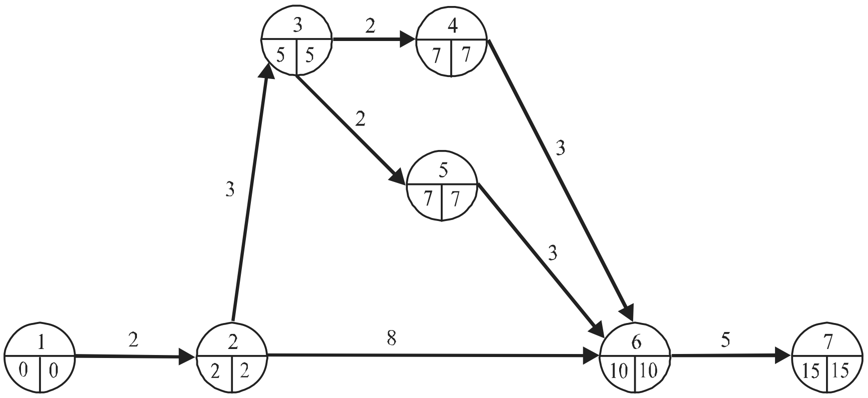


Fig. 10.19

Now we find the all activities have come to the level of being critical with total project duration changing from 16 to 15 weeks and now no other activity can be crashed (Fig. 10.19 refers).

We can now draw the matrix of project duration and total cost of the project.

Activity Crashing	*Project Duration*	*Direct Cost*	*Indirect Cost*	*Total Cost*
None	21	33,000	63,000	96,000
(4-6)	20	34,000	60,000	94,000
(4-6), (2-3)	19	35,000	57,000	92,000
(4, 6), (2, 3), (6-7)	16	39,000	48,000	87,000
(4-6), (2-3), (6-7), (3-4)	15	40,500	45,000	85,500

It can thus be seen that the cost and time considerations on crashing the project duration from 21 to 15 weeks brings out the optimal solution in this form.

The time-cost trade-off, thus, can be achieved when it is possible to expedite the work by increasing resources. It should also be technically feasible to crash the activities. Then only trade-off is effective.

Updating the network based on actual progress of work is required to help in periodic review. CP thus may change on reviews. When there is a time or cost-over-run, the trade-off analysis is an important managerial tool to set-off certain relevant decisions for control function of the project.

10.13 Summary

Important Terms Used for summarising this unit:

- **Concurrent Activities:** Activities of a project that can be performed simultaneously through separate set of resources without disturbing other activities.
- **Most Likely Time:** Amount of time required to complete an activity under real-life situation, with bottlenecks quickly resolved.
- **Optimistic Time:** Shortest time required for an activity when all resources available and no delays expected/occurred.
- **Pessimistic Time:** Longest time required for completion of an activity of a project when misforeseen or unexpected interruption occur during its execution.

- **Earliest Occurrence Time:** An event occurs the earliest, when all the activities leading to it have been accomplished.
- **Latest Occurrence Time:** It represents the latest time by which an event may occur based on the time permitted for the project.
- **Critical Path:** The path on a network travelling through the nodes of zero blacks and representing the longest route on the network.

10.14 Key Words/Abbreviations

- **CPM:** The sequence of stages determining the minimum time needed for an operation, especially when analysed on a computer for a large organization
- **PERT:** Is a statistical tool used in project management, which was designed to analyze and represent the tasks involved in completing a given project.
- **Slack and float:** In project management, float or slack is the amount of time that a task in a project network can be delayed without causing a delay to: subsequent tasks project completion date. Total float is associated with the path.
- **Project completion time:** Project Completion time means the recipient of incentives or assistance has agreed to meet all the terms and obligations contained in an agreement with the authority

10.15 Learning Activity

1. Explain the following terms in PERT/CPM.

   (*i*) Earliest time

   (*ii*) Latest time

(*iii*) Total activity slack

(*iv*) Event slack

(*v*) Critical path

10.16 Unit End Questions (MCQ and Descriptive)

A. Descriptive Types Questions

1. Define “Critical path”, “Slack time”, “Resource Levelling” and “Dummy activity” with reference to PERT and CPM.
2. How Uncertainty can be incorporated in PERT model?
3. Describe Dummy Activity in relation to activity relationships on the network path.
4. Describe Fulkerson’s rule for numbering of activities on the network.

Problems

1. A project plan is as follows:

Activity	*Predecessors*	*Time*	*Activity*	*Predecessors*	*Time*
A	—	8	G	E	6
B	—	2	H	E	3
C	A	1	I	G	3
D	B	9	J	H	5
E	B	4	K	I, J	2
F	C, D	5	L	F	3

Construct a PERT network and compute the early start, late start and slack time for each activity. Indicate the critical path.

2. Draw the network, given the following relationships:

Event Numbers	:	1	2, 3	4	5	6	7
Preceded by	:	—	1	2, 3	3	4, 5	5, 6

3. For a small project of 12 activities, the details are given below. Draw the network and find earliest occurrence time, latest occurrence time, critical activities and project completion time.

Activity : A B C D E F G H I J K L

Dependence : — — — B, C A C E ED, F, H E I, J G

Duration (days) 9 4 7 8 7 5 10 8 6 9 10 2

4. For overhauling of vehicles, following time table is followed in the repair workshop.

Jobs	*Immediate Predecessors*	*Expected Time (days)*
A	—	3
B	A	2
C	B	2
D	B	3
E	C	40
F	C, D	2
G	D	3
H	C	2
I	D	1
J	D	4
K	D	3
L	F	2
M	G	3
N	D, E, L	2
O	N	1
P	N, I, J, M	3
Q	G	2

Draw the network and identify the critical path.

5. A building construction project has the following time schedule.

Activity	*Time in Months*	*Activity*	*Time in Months*
1—2	2	3—7	5
1—3	2	4—6	3
1—4	1	5—8	1
2—5	4	6—9	5
3—6	8	7—8	4
		8—9	3

Construct PERT network and compare (*i*) total float for each activity (*ii*) critical path and its duration. Also find the minimum number of cranes the project must have for its activities 2-5, 3-7 and 8-9 without delaying the project. Is there any change required in the network? If so, mention them.

6. A project has the following characteristics:

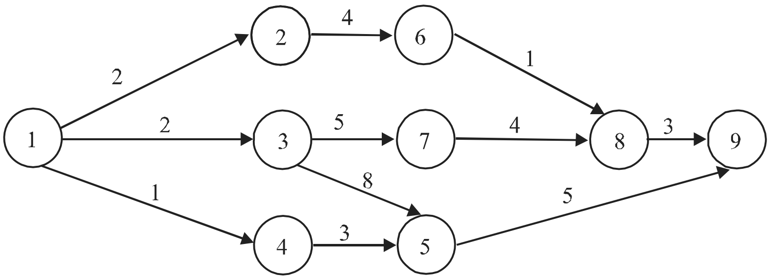
Activity	*Preceding Activity*	*Expected Completion Time in Weeks*
A	None	5
B	A	2
C	A	6
D	B	12
E	D	10
F	D	9
G	D	5
H	B	9
I	C, E	1
J	G	2
K	F, I, J	3
L	K	9
M	H, G	7
N	M	8

(*i*) Draw a PERT network for this project.

(*ii*) Find the various paths and critical path as well as the project completion time.

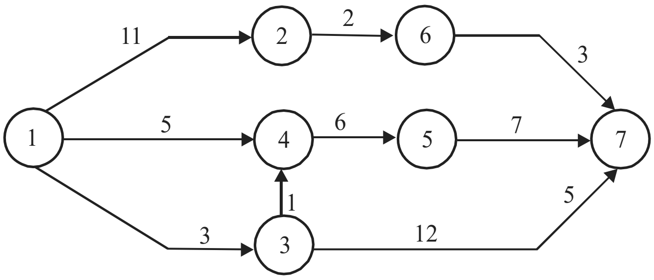
(*iii*) Prepare an activity schedule showing the ES, EF, LS, LF and float for each activity.

(*iv*) Will the critical path change if activity G takes 10 weeks instead of 5 weeks? If so, what will be the new critical path?



7. Give the critical path for the following PERT diagram. Also calculate the slack time for each event

8. The following figure gives a CPM network for a project in arrow notation in which durations



are given in number of weeks. Compute for each job (*i*) Earliest start time (*ii*) Earliest finish time, (*iii*) Latest start time, (*iv*) Latest finish time, (*v*) Total float, (*vi*) Free float (*vii*) Independent float.

9. The following information is known for a project. Draw the network and find the critical path. Capital letters denote activities and numbers in bracket denote activity times

This must be Completed	*Before this can Start*
A (30)	C
B (7)	D
B	G
B	K
C (1)	D

C	G
D (14)	E
E (10)	F
F (7)	H
F	I
F	L
G (21)	I
G	L
H (7)	J (15)
I (12)	J
K (30)	L (15)

10. A project has the following time schedule

Activity	*Time (in months)*	*Activity*	*Time (in months)*
1—2	2	4—6	3
1—3	2	5—8	1
1—4	1	6—9	5
2—5	4	7—8	4
3—6	8	8—9	3
3—7	5		

Construct PERT network. Compute

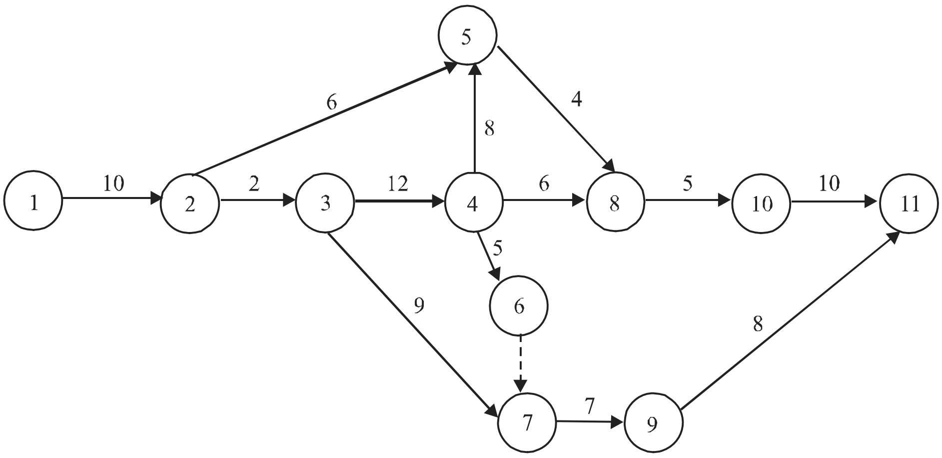
(*a*) Critical path and its duration

(*b*) Total float for each activity

Also find the minimum number of cranes the project must have for its activities 2-5, 3-7 and 8-9 without delaying the project. Then, is any change required in PERT network? If so, indicate the same.

11. Obtain the critical path and project duration for the following PERT network.

12. A project schedule has the following characteristics



Activity	*Time*	*Activity*	*Time*
1—2	4	5—6	4
1—3	1	5—7	8
2—4	1	6—8	1
3—4	1	7—8	2
3—5	6	8—10	5
4—9	5	9—10	7

(*i*) Construct a PERT network.

(*ii*) Compute T_E and T_L for each event.

(*iii*) Find the critical path

(*iv*) Also obtain the total and free floats for each activity.

13. A project is represented by the network shown below and has the following data.

Task	:	A	B	C	D	E	F	G	H	I
Optimistic time	:	5	18	26	16	15	6	7	7	3
Pessimistic time	:	10	22	40	20	25	12	12	9	5
Most likely time	:	8	20	33	18	20	9	10	8	4

Determine the following :

(*a*) Expected task times and their variance.

(*b*) the earliest and latest expected times to reach each event.

(*c*) The critical path.

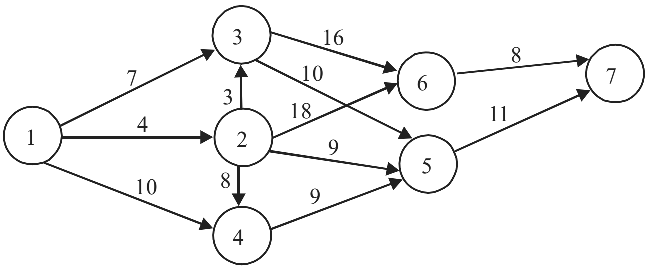
14. For the following project

(*i*) Calculate for each activity, its early start, early finish, late start, late finish, total float and free float.

(*ii*) Identify the critical path.

(*iii*) If the project manager finds that either of the activities 2-6 or 4-5 should be speeded up by 2 days, at the same cost, which of the two activities should be speeded up? Explain.

(*iv*) Assuming that the time estimates in days indicated in the above network represents the expected duration based on three times estimates and suppose the variance along the critical path is 81 days, what is the probability that the project will be completed within 33 days? Within 44 days?



15. The following table gives data on normal time and cost and crash time and cost for a project.

Activity	*Normal*		*Crash*	
	Time (Weeks)	*Cost (₹)*	*Time (Weeks)*	*Cost (₹)*
1—2	3	300	2	400
2—3	3	30	3	30
2—4	7	420	5	580

2—5	9	720	7	810
3—5	5	250	4	300
4—5	0	0	0	0
5—6	6	320	4	410
6—7	4	400	3	470
6—8	13	780	10	900
7—8	10	1,000	9	1,200

Indirect cost is ₹ 50 per week.

(*i*) Draw the network and identify the critical path with a double line.

(*ii*) What are the normal project duration and associated cost?

(*iii*) Find out the total float associated with each activity.

(*iv*) Crash the relevant activities systematically and determine the optimal project completion time and cost.

16. The following is the table showing details of a project

Activity	*Immediate Predecessors*	*Normal*		*Crash*	
		Time (Weeks)	*Cost (₹'000)*	*Time (Weeks)*	*Cost (₹'000)*
A	—	10	20	7	30
B	—	8	15	6	20
C	B	5	8	4	14
D	B	6	11	4	15
E	B	8	9	5	15
F	E	5	5	4	8
G	A, D, C	12	3	8	4

Indirect cost is ₹ 400 per day. Find the optimum duration and associated minimum project cost.

17. The required data for a small project consisting of different activities are given below :

Activity	*Dependence*	*Normal Duration* *(days)*	*Normal Cost* *(₹)*	*Crash Duration* *(days)*	*Crash Cost* *(₹)*
A	—	6	300	5	400
B	—	8	400	6	600
C	A	7	400	5	600
D	B	12	1,000	4	1,400
E	C	8	800	8	800
F	B	7	400	6	500
G	D, E	5	1,000	3	1,400
H	F	8	500	5	700

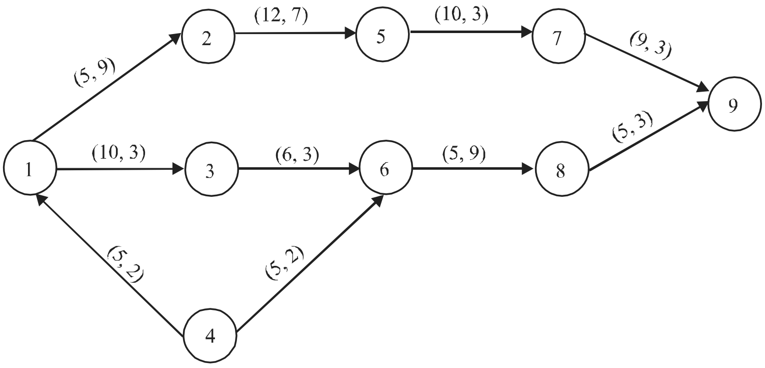
(*i*) Draw the network and find out the normal project length and minimum project length.

(*ii*) If the project is to be completed in 21 days with minimum crash cost, which activities should be crashed by how many days?

18. The President of ABC Manufacturing company has an opportunity to participate in a project that has a sales price of ₹ 90,000 but it must be completed within 8 weeks. Since the President at 8.30 am on Monday (start of the 8 week) has to determine the profitability of the project on an 8 week basis, so that the firm can start the production order at 10 am in order to stay within 8 weeks demanded by the customer whereas the time/cost is based on 10 week basis. A table of times and cost is given below. Did the President accept the project?

Event	*Preceding Event*	*Normal Time* *(weeks)*	*Cost* *(₹)*	*Time* *(weeks)*	*Crash Cost* *(₹)*
4	1	2	8,000	1	13,000
2	1	2	17,000	1	19,000
3	1	6	11,000	5	13,000
4	2	4	6,000	3	10,000
3	2	2	9,000	1	11,000
5	2	7	8,500	6	12,000
5	4	4	10,000	3	16,000
5	3	3	5,000	2	7,000

19. Following is the drawn out project schedule for a small project. The figures given on each arrow indicate the expected time and variance.



Work out the critical path for the project and prepare its bar chart. Based on the variances, indicate which path on the network should be watched carefully and closely?

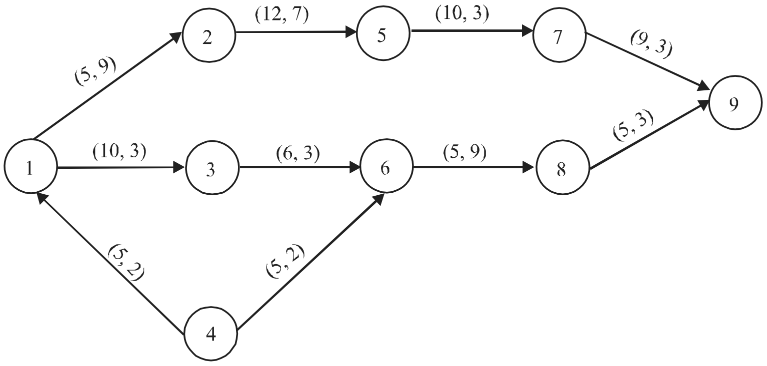
20. Following are the man-power requirements for each activity in a project.

Activity	*Normal Time (days)*	*Manpower Required Per Day*
1—2	10	2
1—3	11	3
2—4	13	4
2—6	14	3
3—4	10	1
4—5	7	3
4—6	17	3
5—7	13	5
6—7	9	8
7—8	1	11

(*a*) Draw the network and find out the total float and free float for each activity.

(*b*) The customer stipulates that during the first 26 days, only 4 to 5 men and during remaining days 8 to 11 men only can be made available. Rearrange the activities suitably for levelling the manpower resources, satisfying the above condition.

21. Given the following network, indicate the critical path.



22. (*a*) Give your comments on the assumptions of PERT and CPM.

(*b*) The following table lists the activities of a maintenance project.

Activity (in months)	*Duration*	*Activity (in months)*	*Duration*
1—2	2	4—7	3
1—3	2	5—8	1
1—4	1	6—8	4
2—5	4	7—8	5
3—6	5	8—9	3
	3—7	8	

(*i*) Draw the project network.

(*ii*) Find the critical path and duration of the project.

(*iii*) Suppose, we are required to employ a special piece of equipment on activities 1-3, 3-6, 2-5, 5-8 and 8-9, one at a time. Will it affect the duration of the project? Explain.

B. Multiple Choice/Objective Type Questions

1. PERT analysis is based on _______________________.

(a) optimistic time (b) pessimistic time

(c) most likely time (d) all the above.

2. The performance of a specific task in CPM, is known ___________________.

   (a) Dummy (b) Event

   (c) Activity (d) Contract.

3. What does the critical path refer to?

   (a) Longest path in terms of time

   (b) Shortest path in terms of time

   (c) Most direct path from the beginning node to ending node

   (d) Path with the largest amount of slack

4. The Optimistic, most likely and pessimistic time estimate for the PERT network of a project are shown in the following table. The expected duration of the project is _________.

Activity	Optimistic Time (to)	Most Likely Time (tm)	Pessimistic Time (tp)
1-2	4	5	6
1-3	2	4	6
1-4	2	3	4

   (a) 36 (b) 37

   (c) 39 (d) 40

5. If the duration of each activity is known with certainty, the Critical Path Method (CPM) can be used __________________________.

   (a) to determine the length of time required to complete a project.

   (b) to estimate the probability that the project will be completed by a given deadline.

   (c) to minimize the total cost

   (d) To maximize the total profit

6. For each activity, PERT requires that the project manager estimate the _______________.
   (a) activity's duration under the most favorable conditions
   (b) activity's duration under the least favorable conditions
   (c) most likely value for the activity's duration
   (d) All of these.
7. You can use the Work Breakdown Structure (WBS) to identify the ________________.
   (a) Minimal Spanning tree problem
   (b) Shortest route problem
   (c) activities involved in the project
   (d) All of these

Answers:

1. (d), 2. (c), 3. (a), 4. (b), 5. (a), 6. (d), 7. (c)

10.17 References

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